14 DEC 1947

THE MATHEMATICS STUDENT

Vol. XIV] MARCH—JUNE 1946 [Nos. 1 & 2

NON DESARGUESIAN GEOMETRY

BY

MISS M. SUSEELA, *Madras University.*

1. The significance of Desargues' theorem

It is well known that Desargues' theorem plays a crucial part in the investigation of the foundations of projective geometry. This is because Desargues' theorem can be proved by the use of the projective axioms of alignment and extension alone, if the dimensionality is greater than two. But curiously enough, it can be shown to be independent of these axioms in the case of dimensionality two, that is, in plane projective geometry. Thus it is to be inferred that the necessary and sufficient condition that Desargues' theorem holds in a plane projective geometry, is that it should be imbeddable, in a higher dimensional geometry. The proof of the theorem may be found in Veblen and Young.[1]

It was Hilbert who established the connection between Desargues' theorem and the algebrisation of geometry. His result was that if Desargues' theorem holds in a plane projective geometry, then the points of that geometry can be represented by pairs of numbers (co-ordinates) chosen from a suitable skew field, so that the equations of straight lines are linear in co-ordinates and the usual processes of analytic geometry can be carried out. [By a skew field Γ is meant a set of elements which satisfy the following conditions:

(1) They form an abelian group w.r.t. addition, so that the identity element, zero, and inverse elements exist uniquely.

(2) The non-zero elements form a group w.r.t. a second operation, multiplication.

(3) Multiplication is distributed both ways by addition.]

It is also known that if in addition, Pascal's theorem holds, the field is commutative.

Conversely, given any skew field Γ, we can define an abstract projective geometry thus—(1) all the ordered number pairs (x, y) of Γ are the points of this geometry; (2) all the single elements of Γ are the points at infinity; (3) all linear equations of the type $ax+by=c$, with coefficients *from* Γ, define the lines of the geometry and all the single elements lie on a particular line, which is the analogue of the line at infinity in ordinary, real projective geometry. We can show that in this projective geometry Desargues' theorem holds and if further the field is commutative, Pascal's theorem also holds. Hence, to sum up, Hilbert has shown that an abstract projective geometry in which Desargues' theorem holds may be algebrised by means of co-ordinates taken from a skew field, in such a way that it is formally of the same sort as an ordinary real projective geometry. The fact that the field is non-commutative does not make a serious difference, and even this non-commutativity disappears if the abstract projective geometry satisfies Pascal's theorem. The general question whether and how far Non-Desarguesian geometry i.e., a plane projective geometry in which Desargues' theorem does not hold, can be algebrised is an interesting problem. The most fundamental result with regard to this question follows from the work of Miss Ruth Moufang.

2. Harmonic Geometry

By a harmonic geometry we mean a plane projective geometry in which the theorem of the complete quadrilateral holds. Let A, B, C be 3 fixed points on a straight line and let the lines AA', BB', through A and B respectively meet at O. Through C a line is drawn cutting these two lines at A' and B' respectively. AB', $A'B$ intersect in O'. OO' meets AB at D. Then the theorem of the complete quadrilateral states that D is a fixed point, whose position is not altered by varying the lines through A, B and C. (D is the "fourth harmonic conjugate" of C w.r.t. A and B in ordinary real projective geometry). The special significance of Ruth Moufang's work is that she has brought into notice a special class of non-desarguesian geometries—viz. those in which the theorem of the complete quadrilateral holds, ensuring thereby, the existence of the fourth harmonic conjugate of any point C w.r.t. two other points A and B collinear with C.

Before stating the main results of Moufang's work it will be useful to notice the special cases of Desargues' theorem with which we will have to deal in the course of our work.

The (p, L) Desargues' theorem states that if 2 triangles in perspective from a point p are such that 2 pairs of corresponding sides intersect on a line L, then the third pair will also intersect on L. Desargues' theorem states that (p, L) Desargues' theorem holds for all points P and all lines L.

By the *Small Desargues' theorem* is meant that (p, L) Desargues' theorem holds for all points p and all lines L, through p.

If 2 perspective Δ^s are such that a side of each passes through a vertex of the other, than the Δ^s are also axial. This may be called the "*doubly specialised*" *Desargues' theorem.*

If 2 perspective Δ^s are such that a side of one passes through a vertex of the other, then the Δ^s are also axial. This is a generalisation of the preceding theorem and may be called "*singly specialised*" *Desargues' theorem.*

Miss Moufang's main results are two in number.

(*a*) The theorem of the complete quadrilateral is equivalent to the Small Desargues theorem.(2)

In order to prove this we make use of the following theorems:—

Theorem 1. The theorem of the complete quadrilateral is equivalent to the doubly specialised Desargues' theorem.

If ABC, $A'B'C'$ are the 2 Δ^s perspective from O, such that BC passes through A' and $B'C'$ passes through A and the corresponding sides BC, $B'C'$ etc. intersect at p, q, r respectively, by the theorem of the complete quadrilateral it is obvious that p, q, r lie in a straight line. Conversely, the doubly specialised Desargues' theorem implies the theorem of the complete quadrilateral.

Theorem 2. From the theorem of the complete quadrilateral follows the singly specialised Desargues' theorem.

The proof according to Moufang is as follows:—

ABC, $A'B'C'$ are the 2 Δ^s perspective from O. The corresponding sides BC, $B'C'$ etc. cut at p, q, r respectively. pA cuts BB', CC' at B'', C'' respectively. Also AB, $A'B''$ cut at l and AC, $A'C''$ at l'. If $C'B'$ cuts $A'A$ at A'', then $A'B'$, $A''B$ and $A'C'$, $A''C$ cut at m, m' respectively. By quadrilateral theorem for the 2 quadrilaterals ($A'A$ BB'') and ($A'A$ CC'') $pl\,l'$ lie in a straight line. Similarly pmm', oml'

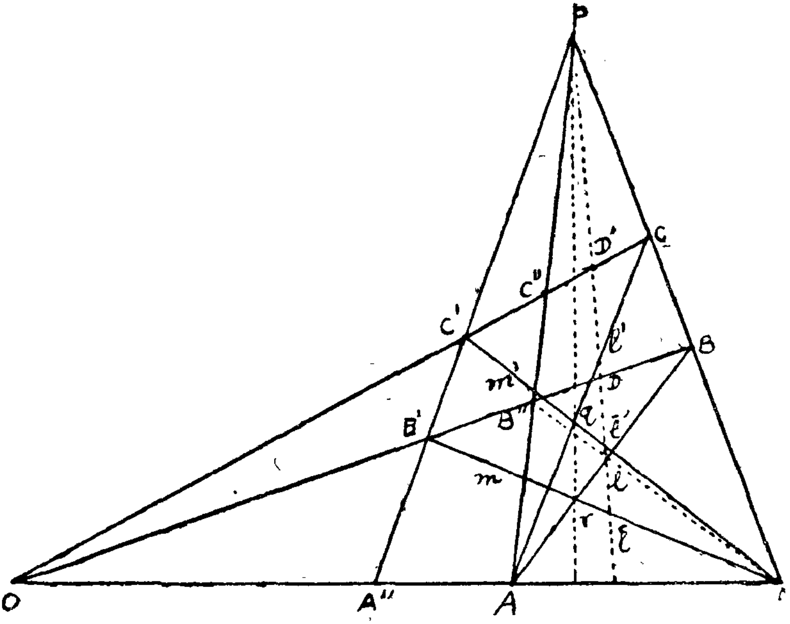


FIG. I

$om'l'$ lie in a straight line. If pl meets the lines BB', CC' at DD, respectively then $ODBB''$ and $OD'CC''$ are sets of harmonic points. Projecting $ODBB''$ from l we have that $m\,\bar{l}\,r\,A$ is harmonic where pl meets $A'r$ at $\bar{l}$. Similarly $m'\bar{l}'qA'$ is harmonic where pl meets $A'q$ at $\bar{l}'$. Projecting $m\bar{l}r\,A$ and $m'\,\bar{l}'\,q\,A'$ into each other from p, we see that rq passes through p. Hence the Δ^s are axial.

Theorem 3. The singly specialised Desargues' theorem implies the following theorem: If two Δ^s BrB', cqc' (Fig. 1) are such that a pair of corresponding sides $B'r$, $c'q$ intersect in A' which lies on a line through the vertices B, C, then the Δ^s are perspective.

This is also seen to be another special case of Desargues' theorem.

Theorem 4. The theorem 3 implies Small Desargues' thorem.

If ABC, $A'B'C'$ are $2\Delta^s$ in perspective from O, where BC, $B'C'$ and AC, $A'C'$ meet at D', D respectively on a line through O, then $A\ B$ and $A'B'$ also intersect on DD', for by theorem 3, for the Δ^s $DA\ A'$, $D'B\ B'$, the lines AB, $A'B'$ and DD' are concurrent.

Conversely it is obvious that the Small Desargues implies theorem 3.

Theorem 5. Theorem 3 implies the doubly specialised Desargues' theorem.

Let ABC, $AB'C'$ be two Δ^s perspective from O, such that A lies on $B'C'$ and A' on BC. The corresponding sides BC, $B'C'$ etc. meet at

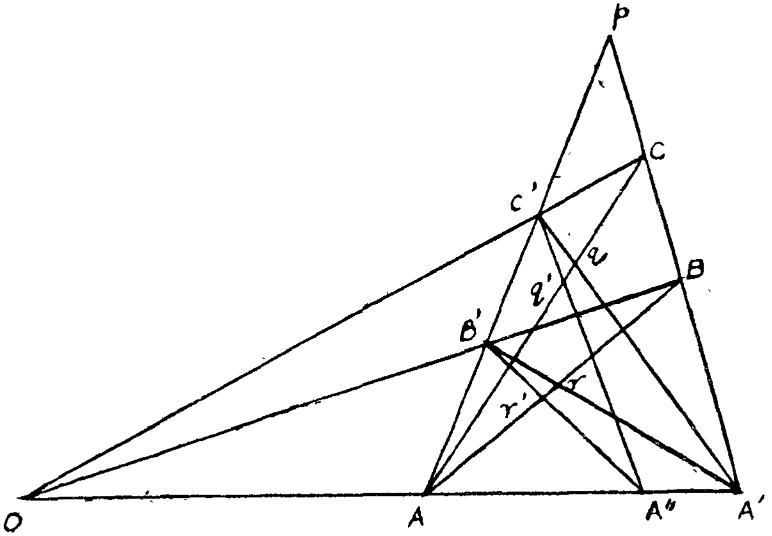


FIG. 2

p, q, r. If A'' be any point on AA', let $B'A$, $C'A''$ cut AB and AC at r', q' respectively. By theorem 3 for the $\Delta^s C'Cq'$ and $B'Br'$, $pq'r'$ lie in a straight line. Similarly for the Δ^s $C'qq'$ and $B'rr'$, pqr lie in a straight line.

Taking into account all the above theorems we see that the theorem of the complete quadrilateral is equivalent to the Small Desargues' theorem.

(*b*) A geometry in which the theorem of the complete quadrilateral holds can be algebrised by means of co-ordinates taken from an alternating field, [3] i.e. a system of elements having the following properties :—

(1) The elements form an additive abelian group.

(2) The system is closed w.r.t. multiplication.

(3) It contains a unit element satisfying $1 \cdot a = a \cdot 1 = a$.

(4) Every element $a \neq 0$, possesses on inverse element a^{-1}, such that $a^{-1} \cdot a = 1 = a \cdot a^{-1}$.

(5) $a^{-1} \cdot (a \cdot b) = b = (b \cdot a)\, a^{-1}$.

(6) Multiplication is both left and right distributive.

From the property (5) follows that $a \cdot (a \cdot b) = (a \cdot a) \cdot b$ and by symmetry also $b \cdot (a \cdot a) = (b \cdot a) \cdot a$ From these two relations it may be shown that $a \cdot (b \cdot a) = (a \cdot b) \cdot a$. The proof of these relations due to Reidemeister has been given by Moufang [3].

Thus the algebrisation of geometry has been carried a step further by Moufang than had been done by Hilbert. A purely algebraical study of these alternating fields has been carried out by Zorn [4]. Now the question arises whether the algebrisation of non-desarguesian projective geometry can be carried still further. This has been done by Baer [5] for a geometry which is not harmonic but in which (p, L) Desargues' theorem for a particular point p, and a particular line L through p holds.

3. Following Baer, we have first to notice the important bearing which (p, L) Desargues' theorem has on the structure of the projective group of the plane. So far as the axioms of the projective plane are concerned they do not imply the existence of projective transformations of the plane into itself, other than the identity. But if (p, L) Desargues' theorem holds there is a group of projective transformations of the plane into itself, which is transitive in a certain sense viz. (p, L) transitive.

A (p, L) perspectivity is defined as a plane collineation (i.e. a transformation taking points into points and lines into lines in a reversible (1, 1) manner) which takes lines through p into themselves and leaves points on L invariant. A plane is said to be (p, L) transitive, if given two points r and s collinear with p, there is a unique (p, L) perspectivity which takes r into s. (p, L) Desargues' theorem is seen to be entirely equivalent to (p, L) transitivity of the plane.

Let $r\, r'\, r''$ and s, s', s'' be the two triangles perspective from p. $r'r''$, $s's''$ meet at D and rr'', ss'' at E on L. If the plane is (p, L) transitive there is a unique perspectivity P taking r' to s'. This perspectivity takes Dr' to Ds' i.e. r'' to s'' and $r''E$ to $s''E$ i.e. r to s. Let rr' cut L in F. Then the perspectivity P takes Fr to Fs and Fr' to Fs'. But Fr and Fr' are one and the same line. ∴ Fs and Fs' should also be in the same line, since there cannot be two trans-

forms of any line in a particular projective transformation. rr' and ss' meet at F on L, ie (p, L) Desargues' theorem holds.

Conversely (p, L) Desargues' theorem can be seen to imply (p, L) transitivity of the plane.

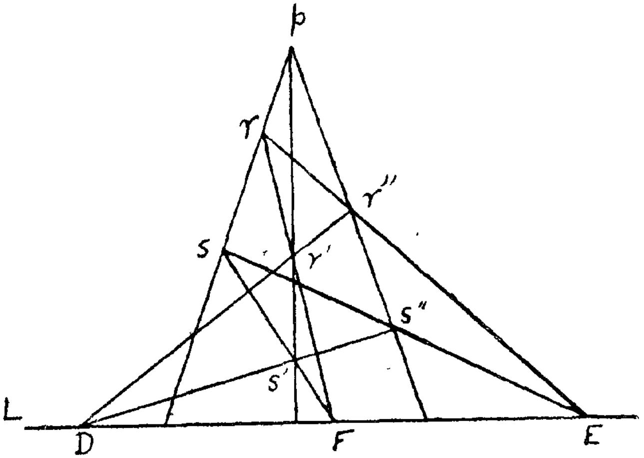


FIG. 3

Let p be any point and L any line. Let r, s be any two points collinear with p. (Fig. 3) Let us define a point transformation $K_{(r,s)}$ thus — $K_{(r,s)}(p) = p$. $K_{(r,s)}$ [any point on L] = the same point. $K_{(r,s)}(r) = s$. If r' is any point not on rs then $K_{(r,s)}(r') = s'$, lying on pr', and such that rr' and ss' meet on L (at F). We see that this transformation has not been defined for points on (r, s). If we define the transformation $K_{(r'\,s')}$ in a similar way then if r'' is any point not on $r's'$, its transform s'' viz $K_{(r',s')}(r'') = s''$ is such that $r'r''$, $s's''$ meet on L (at D). Now by (p, L) Desargues' theorem for the perspective Δ^s $rr'r''$ and $ss's''$, since 2 pairs of corresponding sides intersect on L at D and F respectively, the third pair will also meet on L. Therefore we see that s'' is the corresponding point of r'' for the transformation $K_{(r,s)}$ also. Hence any pair of corresponding points defined by $K_{(r'\,s')}$ will also be corresponding points for $K_{(r,s)}$. The transforms of points on rs, are defined by $K_{(r',s')}$ and hence also by $K_{(r,s)}$. Thus the transformation $K_{(r,s)}$ has been defined for all points of the plane. It is seen at once that such a transformation also takes lines into lines. Hence the transformation is a perspective projective transformation which takes lines through p into

themselves and leaves points on L invariant i.e. a (p, L) perspectivity. Hence we see that if in a projective plane, (p, L) Desargues' theorem holds, the plane is (p, L) transitive. *Thus* (p, L) *Desargues, theorem is entirely equivalent to* (p, L) *transitivity of the plane.*

4. Baer co-ordinate projective geometry

We will now see how co-ordinates can be introduced in a geometry in which (Y, L) Desargues' theorem holds, for a particular point Y and a particular line L through it. X and Z are two other points on L and O is a point not on L. Let F denote the set of all points on OY different from Y. F is itself the algebraic system from which we choose co-ordinates for points of this geometry. The two operations of addition and multiplication in F are derived by purely projective operations.

Taking OX, OY to be analogous to cartesian axes and L to the line at infinity, a system of co-ordinates may be derived for all points not on L. If P is any point not on L, PY is joined to cut OX in α_1. $Z\alpha_1$ cuts OY in α. PX cuts OY in β. Then (α, β) are said to be the co-ordinates of P. Conversely given (α, β) the point P is uniquely determined.

If a and b are any two points of OY the line Xb is denoted by $y=b$. The line Za will meet OX in a point not on L The join of Y with this point is denoted by $x=a$. It is seen that $x=a$, $y=b$ meet in the point whose co-ordinates are (a, b).

Addition in F is defined as follows:—if α and β are any 2 points in F, then $\alpha+\beta=P_\beta(\alpha)$ where P_β is the unique (Y, L) perspectivity taking O to β. Since the (Y, L) perspectivities form a group, the elements of F form a group w.r.t. addition, which is not, in general, commutative. The inverse elements and the identity element, seen to be O itself, exist uniquely.

In order to define multiplication, an arbitrarily chosen element in F is called 1 and the product of the numbers (r, s) is so defined that (r, rs) is the intersection of $x=r$ with the join of 0 and $(1, s)$. The join of $(0, 0)$ with $(1, s)$ is seen to have the equation $y=xs$. To derive the equation of any line, not through O, we observe that any such line can be carried into a line through O by a (Y, L) perspectivity. For, if such a line R cuts OY in $(0, r)$ then there is a unique perspectivity P_{-r} taking $(0, r)$ to $(0, 0)$. Then $P_{-r}(R)$ is $y=xs$ (say), so that $R=P_r$ { Join of $(0, 0)$ and $(1, s)$ } i.e $y=xs+r$. All lines are seen to have equations of this type except those through

Y which have equations $X=$ constant. It is obvious that two lines of the type $y=xs+r$ and $y=xs+r'$ intersect only on L, since they can each be regarded as transforms of $y=xs$. Two such lines with equal coefficients of x, correspond to parallel lines in ordinary Euclidean geometry and s corresponds to the gradient. The points on L correspond reversibly to the coefficients s. X and Y are denoted by L (0) and L (∞) respectively since they correspond to zero and ∞ on L. Thus, to sum up according to Baer the elements in F have the following properties.

(1) They form a group w.r.t. addition which is, in general non-commutative.

(2) The system of elements in F is closed w.r.t. multiplication which need not be commutative, associative or distributive but is such that:

(2·1) $0\cdot r=r\cdot 0=0$

(2·2) there exists an element $1\neq 0$, satisfying $1\cdot r=r$, $r\cdot -1=-r$ for all r in F.

(2·3) given r, s, t ($r\neq s$) there exists a unique x and a unique y in F such that $-xs+xr=t$, $sy-ry=t$.

A system of numbers satisfying the above properties may be called a 'Baer Algebra'. Conversely given a Baer Algebra we can construct abstractly a geometry in which (p, L) Desargues' theorem for a particular point p and a particular line L, through it holds, and such a geometry may be called 'Baer Geometry.' To the best of our knowledge Baer's geometry is the most general non-desarguesian geometry in which we have workable system of co-ordinates. I have been interested in examining the steps by which Baer geometry specialises into more restricted systems of non-desarguesian geometry. I shall give, below a few of my results.

5. Dickson Geometry

(a) In generalising the notion of finite fields Dickson appears to have been led to an important class of finite algebras, each consisting of a finite number of elements. The elements of any particular finite algebra satisfy the following properties :—

(1) They form a group w.r.t. addition, which is non-abelian. The identity and inverse elements exist uniquely.

(2) The non-zero elements form a group w.r.t. multiplication which is non-commutative. The identity element and the inverse elements of all non-zero elements exist uniquely.

(3) If a, b, c are elements of the algebra, then $a(b+c)=ab+ac$.

A system of elements satisfying these properties may be called a 'Dickson Algebra'. Given an algebra with these properties we can construct an abstract geometry which may be called "Dickson Geometry" since Dickson appears to have been the first person to construct finite Geometries of this kind, as is seen in Carmichaels' "Theory of Groups of Finite Order." (6)

I shall now show that this Dickson Geometry is a special case of Baer geometry which is derived by the assumption of (O, L) Desargues' theorem, in addition to (Y, L) Desargues' theorem. For, when (O, L) Desargues' theorem is also assumed the number system of the Baer Algebra develops two new properties, left distributivity and associativity of multiplication.

Since (O, L) Desargues' theorem holds i.e., the plane is (O, L) transitive, there is a perspectivity P_r taking $(1, 0)$ to $(r, 0)$. Then $P_r(x=1)$ is $x=r$.

$P_r(1, s)=P_r$ (intersection of $x=1$ with $y=xs$) = intersection of $x=r$ with $y=xs$ i.e., (r, rs).

$$P_r(y=s)=P_r\{\text{join of } L(0) \text{ and } (1, s)\}$$
$$=\{\text{join of } L(0) \text{ and } (r, rs)\} \text{ i.e., } y=rs.$$

$P_r(x=s)=P_r\{\text{join of } L(\infty) \text{ and } (s, -s)\}=\{\text{Join of } L(\infty) \text{ and } P_r(s, -s)\}$
$(s, -s)$ is the intersection of $y=-x$ and $y=-s$.

$\therefore$ $P_r(s, -s)=\{\text{intersection of } y=-x \text{ and } y=r\cdot -s\}=\{-(r\cdot -s), (r\cdot -s)\}$

Hence $P_r(x=s)$ is $x=-(r\cdot -s)$, so that $P_r(x, y)=\{-(r\cdot -x), r\cdot y\}$

$P_r(y=xs+t)=P_r\{\text{join of } L(s) \text{ and } (0, t)\}=\{\text{join of } L(s) \text{ and } (0, r\cdot t)\}$
$=y=xs+rt.$

$(1, s+t)$ is on the line $y=xs+t$; $\therefore$ $r, r(s+t)$ is a point on the line $y=xs+rt$

$$r(s+t)=rs+rt.$$

$P_r(x, xt)=\{-(r\cdot -x), r\cdot xt\}$ since $y=xt$ is an invariant line.

$P_r(x, xt)$ should be of the form $(x_1, x_1 t)$

$\therefore$ $\{-(r\cdot -x)\}\, t=r\cdot xt.$

But the distributive law implies that $r\cdot(-x)=-rx$

$$\therefore \quad rx\cdot t=r\cdot xt.$$

Now we see that the property (5) of Baers' Algebra (viz. given r, s, t $r\neq s$ there exists a unique x and a unique y in F such that

14 DEC 1947 MADRAS

$-xs+xr=t,\ sy-ry=t,$) implies the existence of both the left and right inverses w.r.t. multiplication, of all the non-zero elements for suitable values of r, s, t. Further the associative law shows that $a\cdot 1=a$ for any element $a\neq 0$. Since $a\,(1\cdot b)=(a\cdot 1)\,b$ i.e., $a\cdot b=(a\cdot 1)\,b$ hence $a=a\cdot 1$. Thus we see that by the addition of associativity the elements of Baer algebra ($\neq 0$) form a group w.r.t. multiplication and by the further addition of left distributivity Baer Algebra becomes a Dickson Algebra. Hence Dickson Geometry is a special case of Baer Geometry.

We have seen that (O, L) transitivity implies left distributivity and associativity of multiplication. Conversely assuming these two laws we can show that the plane is (O, L) transitive.

Consider the point transformation taking (x, y) into (rx, ry) and leaving points on L invariant. It takes $(x, xp+q)$ into $\{rx, r\,(xp+q)\}$ i.e., $\{rx,\ rx\cdot p+rq\}$ (by the left distributive and associative laws). If $Y=Xp+q$ is any line, it consisits of all points of the type $x,\ xp+q$, the point p on L and $(0, q)$ on OY. The points $(x, xp+q)$ are carried into $(x_1, x_1\,p+rq)$; p is invariant and $(0, q)$ is carried into $(0, rq)$. Now we see that all these points lie on a line viz. $Y=Xp+rq$. $\therefore$ the transformation carries lines into lines i.e., it is a projective transformation carrying lines through p into themselves and leaving points on L invariant i.e., it is a (O, L) perspectivity. Hence the plane is (O, L) transitive.

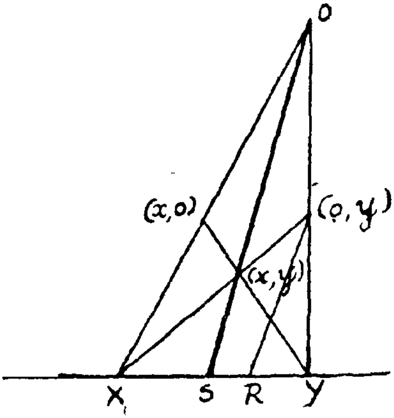


FIG. 4

(*b*) If we assume (Y, OY) Desargues' theorem or (Y, OY) transitivity we can show that the left distributive law holds. Since lines through Y are left invariant (x, y) becomes (x, y'). There is

a unique perspectivity P_r which takes X to R on L where R is given by the totality of lines $Y=Xr+m$ for varying values of m. The line joining X to (x, y) cuts OY in $(0, y)$ which is invariant. By P_r this line goes over into the line joining R to (x, y') and since it should pass through $(0, y)$ its equation is $Y=Xr+y$. Since (x, y') lies on it $y'=xr+y$. The point (x, xp) will be carried into $(x, xr+xp)$ by P_r. Suppose the line $Y=xp$ becomes the line $Y=x\cdot(q)$. When $x=1$, $(1, p)$ becomes $(1, r+p)$ i.e., the line $Y=p$ becomes $Y=r+p$. $\therefore$ $1\cdot q=r+p$. i.e., $q=r+p$. Hence

$$x(r+p)=xr+xp.$$

Conversely assuming left distributivity we can show that (Y, OY) transitivity holds just as in the previous case, by considering the point transformation taking (x, y) to (x, y') where $y'=xr+y$ and the points m on L to $r+m$.

Thus we find that intermediate between Baer and Dickson geometry we have this $\{(Y, L), (Y, OY)\}$ geometry.

(c) A third case of Baer geometry is got by assuming the plane to be (X, L) transitive in addition to being (Y, L) transitive. Since lines through X are left invariant (x, y) is carried into (x', y). There is a unique perspectivity P_r taking $(0, 0)$ to $(r, 0)$. If the line joining $(0, 0)$ to (x, y) cuts L in the point S where OS has the equation $Y=Xs$, S is an invariant point. P_r takes OS into the join of $(r, 0)$ and S i.e., into the line $Y=Xs-rs$. Since (x', y) lies on this line $y=x's-rs$ or $x's=y+rs$ i.e, $xs+rs$ (since $y=xs$). By P_r, OY becomes the join of $(r, 0)$ and Y i.e., $X=r$.

$\therefore$ $P_r(0, k)=(r, k)$. The line joining Z to $(x, 0)$ cuts OY in $(0, x)$ which goes over into (r, x) by P_r.

$\therefore$ the equation of line through Z and (r, x) is $X+Y=r+x$. Since $(x', 0)$ lies on it $x'=r+x$.

$\therefore$ $x's=(r+x)s=xs+rs$. i.e., we get the inverted right distributive law.

Another important result in this case is that addition is commutative. Since the plane is both (X, L) and (Y, L) transitive it is (XY, L) transitive i.e., it is (p, L) transitive for every p on XY. Let Y_r be the (Y, L) perspectivity taking $(0, 0)$ to $(0, r)$ and X be the (X, L) perspectivity taking $(0, 0)$ into $(s, 0)$.

Then $X_s\, Y_r\, (x, y)=(s+x, y+r)$.

Suppose the equation of line joining $(0, 0)$ and (s, r) is $Y=Xt$, then $r=s\cdot t$.

Let this line cut L in T. Then the product of the perspectivities X_s and Y_r (which is also a perspectivity), is the (T, L) perspectivity taking $(0, 0)$ to (s, r). Consider any line through $T - Y = Xt + m$. If (x, y) is a point on it, then $(s+x, y+r)$ should also lie on this line.

Hence we have $y = xt + m$ ... (1)

$$y + r = (s + x)\, t + m$$

i.e. $y + r = xt + st + m$ ... (2)

From (1) and (2) $y + r - y = xt + st - xt$

i.e. $-xt + y + st - y + xt = st.$

If $-xt + y = k$ and $st = k'$, then

$$k + k' - k = k'$$

$$\therefore \quad k + k' = k' + k.$$

Hence addition is commutative.

6. Examples of non-desarguesian geometry got by deforming Euclidean lines

It would be interesting to see whether we can construct non-desarguesian geometries of the foregoing types by deformations of straight lines of the Euclidean plane. Moulton has given a simple example of this type in his paper on "Simple non-desarguesian and non-pascalian geometries" [7] This appears to be a case of "Baer geometry". In Hilberts' "Foundations of Geometry" we find another such example of non-desarguesian geometry. Whether we can systematically construct such geometries by deformations of straight lines of the Euclidean plane, so that we have examples of the various types of non-desarguesian geometry would be an interesting investigation.

References

(1) Veblen and Young—*Projective Geometry* Vol. 1 p. 41.

(2) R. Moufang—*Mathematische Annalen* Bd. 106, 1932, p. 763.

(3) R. Moufang—*Abhandlungen Math. Seminar Hamburg* Bd. 9, 1933, p. 207.

(4) M. Zorn—*Abhandlungen Math. Seminar Hamburg* Bd. 8, 1931, p. 395.

(5) R. Baer—*Transactions of the Amer. Math. Society* Vol. 64, 1942, p. 137.

(6) R. D. Carmichael—*Theory of Groups of Finite Order* p. 396.

(7) F. R. Moulton—*Transactions of the Amer. Math. Soc.* Vol. 3, 1902, p. 192.

THE LAW OF LARGE NUMBERS

BY

D. D. KOSAMBI, *Tata Institute of Fundamental Research, Bombay.*

1. Introduction. Let $X_1, \ldots\ldots, X_n$ be a sequence of random (stochastic) variables with expectations $E(X_r) = m_r$. Then, LLN* states that under suitable restrictions upon the X_r, the difference $(X_1 + \cdots + X_n)/n - (m_1 + \cdots + m_n)/n$ may be made arbitrarily small in absolute value with probability arbitrarily close to unity by taking n sufficiently large (U, chap. 10). The proof is based upon Tshebysheff's Lemma (U, p. 182): *If X is a positive random variable, then*

$$P\{X \leqslant t^2 E(X)\} > 1 - \frac{1}{t^2}.$$

Thus, if $B_n = E(X_1 - m_1 + \cdots + X_n - m_n)^2$ exists, the lemma applied to the positive random variable $\sum_1^n (X_r - m_r)^2$ gives at once

$$\left| \frac{X_1 + \cdots + X_n}{n} - \frac{m_1 + \cdots + m_n}{n} \right| \leqslant \varepsilon \text{ with } P_n > 1 - \frac{B_n}{n^2 \varepsilon^2}.$$

So, for convergence in probability, it suffices that $B_n/n^2 \to 0$. On the other hand, if for $r = 1, 2, \ldots, n$, $\max |X_r - m_r| \leqslant C_n < \infty$, and LLN holds, so that $(1 - P_n) \to 0$, we may use the easily derivable inequality $B_n < n^2 C_n^2 (1 - P_n) + n^2 \varepsilon^2 P_n$ to prove that $B_n/n^2 \to 0$ follows as a necessary condition from LLN under the restriction $C_n^2 (1 - P_n) \to 0$; hence, in particular when the variables X_r are uniformly bounded, $C_n \leqslant C$ (U, pp. 185–6). This may be and has been extended in various directions, beginning with the Bienayme-Tshebysheff inequality (Cr. pp. 21; 38–39). What interests us here is the analysis of the structure of the proof, using only text book methods as far as possible.

In what follows I consider LLN *only* in the particularly useful special case when the variable X_r are all independent.

2. The law of large numbers. To cover general types of distributions, we assume at the outset that each X_r has a distribution function $F_r(x)$ which is positive, non-decreasing, with $F_r(-\infty) = 0$, $F_r(+\infty) = 1$ for all r and $P\{X_r \leqslant x\} = F_r(x)$ for all values

* Abbreviations used are: LLN for the law of large numbers; cf. for characteristic function; U for J.V. Uspensky, "Introduction to Mathematical Probability" (New York, 1937); Cr. for Harald Cramér, "Random Variables and Probability Distributions" (Cambridge Tract no. 36, Cambridge, 1937).

of the real variable x. The integral being taken in the sense of Lebesgue-Stieltjes, we may assume the existence of $E(X_r)=\int x dF_r(x)$ over the entire real line $-\infty \leqslant x \leqslant \infty$. Otherwise LLN has to be given a special meaning for the occasion. We make the further assumption that $\int |x| dF_r(x)$ also exists for each value of the index; this is "reasonable" in that when the distribution is given only discrete values for the observable variable, we have an infinite series replacing the integral and in view of the fact that a preferential order is hardly reconcilable with the intuitive idea of randomness, the series in question would have to converge absolutely, i. e. independently of order, to the same value.

These fundamental assumptions are then also fulfilled for the independent stochastic variable $X_r - E(X_r)$ with which alone LLN is concerned, so that there is no further loss of generality in assuming $E(X_r)=0$ for all r. We are therefore dealing with a sequence of random independent variables $X_1, \ldots, X_n$ such that $\int x dF_r(x)=0$ for all r, and $\int |x| dF_r(x)$ exists. LLN holds for these if and only if $(X_1+\cdots+X_n)/n$ converges in probability to zero. For each n and N, we define non-negative functions of the two variables n, N (of which the first is only a positive integer, the second a continuous variable) as follows:

$$(1) \quad \int_{|x|>N} |x| dF_n(x) = h(n, N) \to 0 \text{ as } N \to \infty \text{ for each } n,$$

$$h_0 = \max h(r, N), \; r=1, 2, \ldots, n; \; H=\sum_1^n h(r, N).$$

$$\int_{|x|\leqslant N} |x| dF_n(x) = c(n,N); \; c_0 = \max c(r,N), \; r=1, 2, \ldots, n; C=\sum_1^n c(r,N).$$

Following a standard device (U, p. 192) the variables X are each split up into two additive stochastic components as follows:

$$(2) \quad X_i = u_i + v_i; \text{ if } |X_i| \leqslant N, \; u_i = X_i, \; v_i = 0; \text{ otherwise } u_i = 0, \; v_i = X_i.$$

We then define $b_i = E(u_i) = -E(v_i)$ and it follows that

$$\begin{aligned}|X_1+\cdots+X_n| &\leqslant |u_1+\cdots+u_n| + |v_1+\cdots+v_n| \\ &\leqslant |u_1-b_1+\cdots+u_n-b_n| + |b_1+\cdots+b_n| \\ &\quad + |v_1+\cdots+v_n|.\end{aligned}$$

By definition it also follows that

$$(3\text{—a}) \quad |b_n| = \Big|\int_{|x|>N} x dF_n(x)\Big| \leqslant h(n, N); \; |b_1+\cdots+b_n| \leqslant H(n, N).$$

$$\text{b)} \quad P\{v_n \neq 0\} = P\{|X_n|>N\} = \int_{|x|>N} dF_n(x)$$

$$= \frac{N}{N}\int\limits_{|x|>N} dF_n(x) \leqslant \frac{1}{N}\int\limits_{|x|>N} |x|\, dF_n(x),$$

whence $P\{v_n \neq 0\} \leqslant \frac{h(n, N)}{N}$ and $\Sigma P\{v_r \neq 0\} \leqslant H(n, N)/N$.

From (2) and the above, we have

$$(4)\quad P\left\{\frac{|X_1 + \cdots + X_n|}{n} \leqslant \varepsilon\right\}$$

$$> P\left\{\frac{|u_1 - b_1 + \cdots + u_n - b_n|}{n} \leqslant \varepsilon - \frac{H(n,N)}{n}\right\} - \frac{H(n, N)}{N}.$$

The argument is that $|X_1 + \cdots + X_n| \leqslant A$ may hold in two mutually exclusive ways: when all the $v_r = 0$, or when at least one $v \neq 0$. The probability for the latter event is allowed for by subtracting the term H/N, which it can never exceed.

The ε in (4) may be chosen arbitrarily small, so that H/n must tend to zero with increasing n. For LLN to hold, H/N must also tend to zero with increasing N, for some method of having n, N both $\to \infty$. A third condition has to be satisfied, however. The stochastic variables $u_i - b_i$ are independent with zero expectation each, and bounded so that their second moment exists; the second moment of their sum is the sum of their second moments. It is easily proved that, for any random variable, $E(X^2) \geqslant E[X - E(X)]^2$, so that $E[\Sigma (u_i - b_i)^2] = B_n \leqslant \Sigma E(u_i^2) \leqslant N\Sigma E(|u_i|)$, whence $B_n \leqslant NC$.

Applying LLN in its classical form to $u_i - b_i$, we obtain

$$(5)\quad P\left\{\frac{|u_1 - b_1 + \cdots + u_n - b_n|}{n} \leqslant \varepsilon - \frac{H}{n}\right\} > 1 - \frac{NC}{n^2(\varepsilon - H/n)^2}.$$

That is,

$$(6)\quad P\left\{\frac{|X_1 + \cdots + X_n|}{n} \leqslant \varepsilon\right\} > 1 - \frac{NC}{(n\varepsilon - H)^2} - \frac{H}{N}.$$

So, for *LLN* to hold, the third condition is that $NC/(n\varepsilon - H)^2 \to 0$. In this limit, the quantity $\varepsilon - H/n$ may be ignored, which gives our main result: LLN *holds*, $(X_1 + \cdots + X_n)/n$ *converging in probability to zero, when* $E(X_r) = 0$ *and* $E(|X_r|)$ *exists for all* r, *if for some manner of approach of* n *and* N *to infinity the conditions*

$$(7)\quad \frac{1}{n}\sum_{1}^{n}\int\limits_{|x|>N} |x|\, dF_r(x) \to 0 \quad ; \quad \frac{1}{N}\sum_{1}^{n}\int\limits_{|x|>N} |x|\, dF_r(x) \to 0,$$

$$\frac{N}{n^2}\sum_{1}^{n}\int\limits_{|x|\leqslant N} |x|\, dF_r(x) \to 0$$

are all satisfied.

In particular, as a corollary, LLN *holds if* $h\ (n, N) < G(N) \to 0$. In this case we need not attempt refinement by asking the condition to hold for all large n in view of the fact that $h\,(n, N) \to 0$ for each n as $N \to \infty$; so that if the condition holds for $n \geqslant k$, the larger of the functions $h_0\,(k, N)$ and $G\,(N)$ will do for all n. The proof of this corollary is simple: the condition leads at once to $H(n, N) < nG\,(N)$, $H/n < G\,(N) \to 0$; $H/N < \eta \sqrt{G} \cdot \sqrt{G}/N \to 0$ also, if we take $n = N/\sqrt{G} \to \infty$.

One implication of this corollary is that *the absolute expectations* $E(|X_r|)$ *are bounded.* For,

$$E(|X_n|) \leqslant N + h_0 < N + G(N) < N + \varepsilon$$

for some N with ε as small as desired, and for all n. The corollary includes as special cases

1) Markoff's: LLN *holds if* $E(|X_r|^{1+\delta})$ *exist and are bounded for all* r *and some* $\delta > 0$. In this case, again, no loss of generality is caused by taking $E(X_r) = 0$. Therefore,

$$\int |x|^{1+\delta}\, dF_r(x) < A \text{ leads to } h < A/N^{\delta} = G\,(N).$$

2) Khintchine's LLN *holds if all the* X_r *have the same distribution and* $E\,(|X|)$ *exists.* In this case, for all n,

$$h = \int_{|x|>N} |x|\, dF(x) = G\,(N) \to 0,$$

taking $E(X) = 0$ as before. Our result is only a slight and almost obvious refinement of existing deductions. The basic process (due apparently to Markoff is the division of the variables into two portions, of which one is bounded and the other contains values of negligible probability. The question still remains: what is the function of the *second* moment for the bounded part in the deduction?

3. The law of large numbers and the central limit theorem. For a simple random variate X, bounded with $|X| \leqslant M$ and zero expectation, the probability of X lying outside a given interval centered at the origin can be increased by increase of the dispersion. Under the conditions of the problem, there is an optimum "most scattered" distribution, independent of the particular and variable limits outside which X may be asked to lie from stage to stage; that is, $X = \pm M$ with probability $\frac{1}{2}$ each will give the greatest possible scattering once for all. Then $P\{|X| > S\} = 0$ if $S \geqslant M$ and $= 1$ if $S < M$. If the value of S be preassigned and not greater than M, we have some choice in limiting the random variable, but the distribution given here is the optimum in that it gives the greatest probability, independently of S.

For two independent variables of the same type, the sum $X_1 + X_2$ can be dispersed in a similar manner. Here, $P\{|X_1 + X_2| > S\} = 1, \frac{1}{2}$, or 0 according as $S < M$, $M \leqslant S < 2M$, or $S \geqslant 2M$. The first can be obtained in particular by taking one of the variables $\pm M$ with probability $\frac{1}{2}$ each and the other 0 with probability 1. In the last case no permissible choice of distributions can possibly give any other probability than zero. In the middle case, however, there does exist an optimum, i.e. X_1, $X_2 = \pm M$, $p = \frac{1}{2}$ each, so that the value of the sum would be $\pm 2M$ with $p = \frac{1}{4}$ each and 0 with $p = \frac{1}{2}$. This, incidentally, points out the essential reason for the validity of our LLN, in that the values of the independent stochastic variables with zero expectations cancel out. But it is of basic importance for the optimum to exist, in such addition, that the interval $2S$ be sufficiently large. For the purpose of the preceding section, it suffices to note that S must lie between $(n-1)M$ and nM to enforce the optimum scattering for the sum of n variates. In the variables $u_t - b_t$, S corresponds to $n\varepsilon - H$, and M to N, or rather to $N + h_0$. So, N must be of the same order as $\varepsilon - H/n$, which will not do in view of the fact that ε is arbitrarily small while N has to be taken arbitrarily large.

If we take the distribution with optimum scattering, $X_i = X = \pm M$, with $p = \frac{1}{2}$ each, then each X has the c.f.

$$\tfrac{1}{2}(e^{-iMt} + e^{+iMt}) = \cos Mt = 1 - 2\sin^2 \tfrac{1}{2} Mt.$$

The c.f. of the sum of n is therefore $(1 - 2\sin^2 \frac{1}{2} Mt)^n$, which, for the average of n tends to $(1 - M^2t^2/2n^2)^n \to \exp(-M^2t^2/2n)$. So the general distribution, bounded by the most scattered case, is thus bounded by a normal distribution with zero mean and variance M^2/n, in substance. If M and N are of the same order, we must have $(N + h_0)^2/n \to 0$ simultaneously with $n\varepsilon - H \sim n(N + h_0)$; but in LLN it is essential to have $N \to \infty$ unless we restrict ourselves to almost trivial cases. These conditions obviously contradict each other.

The function of the second moment, or any moment higher than the first, is to bridge this gap between the requirements of LLN and the approach of this section, by consideration of the most scattered variables.

Suppose that to our preceding assumptions $E(X) = 0$, $|X| \leqslant M$, we add the condition $E(X^2) \leqslant kM$. In the former case, $E(X^2) = M^2$ for the most scattered distribution, so that for $k < M$ (*which we assume hereafter*), we have less concentrated scattering. The extreme limits $\pm M$ can no longer be attained as before with $p = \frac{1}{2}$. The greatest concentrated scattering is, in fact, now given by $\pm \sqrt{kM}$, $p = \frac{1}{2}$ each.

Nevertheless, $P\{|X| \geqslant S\}$ need not vanish if we push S beyond $\sqrt{kM}$ so that the actual optimum in this case is much less clear-cut. For $M > S \geqslant \sqrt{kM}$, the best choice is clearly to take $X = \pm S$ with probability $kM/2S^2$ for each value, and $X=0$ with probability $1-kM/S^2$. If the probability is reduced at either of the two extremes, it would be necessary to add an extra value on the same side of zero, or to cut down the probability for the other extreme, in order to preserve the mean value $E(X)=0$. The optimum for this three-valued distribution is more clearly dependent upon the choice of interval, and the probabilities for the greatest scattering also depend upon S. The c. f. for a single such scattered variable is $1-(2kM/S^2)\sin^2\frac{1}{2}St$. For the sum of n, we raise this to the nth power, for the average of n, we have only to replace t by t/n in raising to nth power. Similarly, for the sum of n, the second moment is $B_n \leqslant nkM$; for the average, we replace each X by X/n and B_n by $B_n/n^2 \leqslant kM/n$. So, in getting the distribution of the average of n values we should not only raise the c. f. to the nth power but also replace t by t/n and S by $S/\sqrt{n}$. Making these substitutions, we get the most scattered distribution of the average as having the c. f.

$$(8) \quad \left(1-\frac{2kMn}{S^2}\sin^2\frac{St}{2n\sqrt{n}}\right)^n \to \left(1-\frac{kMt^2}{2n^2}\right)^n \to e^{-kMt^2/2n}.$$

In spite of its fundamental rôle, S has cancelled out in the process of deduction, to give a normal distribution for the bounding scattered variate. The mean is zero as before, the variance reduced to kM/n. This must tend to zero if LLN is to hold. For the attack of section 2 we should have to take $M=N+h_0$, $k=c_0$, $S\sim\sqrt{c_0 nM}=n(\varepsilon-h_0)$. The condition would then take on the form $h_0 \to 0$, $c_0(N+h_0)/n \to 0$, which it is possible to fulfil, as for example under the assumptions of the preceding section.

That is, the second moment or any moment higher than the first helps only by restricting the admissible variation, the maximum possible scattering. The analysis of this section shows in addition to this a general connection between LLN and the central limit theorem. In fact, we have the limiting values of the bounding distributions as normal distributions.

ON THE CAYLEYAN OF A NON-SINGULAR CUBIC

BY

B. K. SEN GUPTA, *Srikhanda, Bengal*

1. *The equation of the Cayleyan in trilinear co-ordinates.*

Let the equation of a non-singular cubic be $x^3+y^3+z^3+6mxyz=0$; then the equation of its Hessian is

$$m^2(x^3+y^3+z^3)-(1+2m^3)\,xyz=0.$$

Let A and B be two conjugate poles on the Hessian which is cut by the line AB again in C. The tangents at A and B to the Hessian intersect on it at D which is the pole conjugate to C. From D two other tangents can be drawn to the Hessian, and if P and Q be the points of contact, the line PQ must also pass through C; and AB and PQ together form the polar conic of D with respect to the given cubic, P and Q being also a pair of conjugate poles. The Cayleyan is the envelope of AB and if AB is the line $\alpha x+\beta y+\gamma z=0$, the tangential equation of the Cayleyan is found to be

$$m(\alpha^3+\beta^3+\gamma^3)+(1-4m^3)\,\alpha\beta\gamma=0.$$

Put $1-4m^3=3mm'$ and this equation becomes $\alpha^3+\beta^3+\gamma^3+3m'\alpha\beta\gamma=0$. Just as a tangent to a curve is formed by joining two consecutive points on it and taking the limit, so a point on a curve is obtained by taking the limit of the intersection of two consecutive tangents to it. By eliminating α between the equation of AB and that of the Cayleyan, we have

$$\beta^3(x^3-y^3)-3\beta^2\gamma y\,(m'x^2+yz)-3\beta\gamma^2 z\,(m'x^2+yz)+\gamma^3(x^3-z^3)=0.$$

The vanishing of the discriminant of this equation in β and γ is the condition that two of the tangents to the Cayleyan will coincide in the limit, the point of contact of the coincident tangent being a point on the Cayleyan; so that condition gives the equation of the Cayleyan to be in the variables x, y, z,

$$[(x^3-y^3)(x^3-z^3)-yz\,(m'x^2+yz)^2]^2=4x^4\,(m'x^2+yz)^2\,(m'y^2+zx)\,(m'z^2+xy)$$

containing the factor x^6; hence rejecting it, we shall get the equation of the Cayleyan to be of degree six. This equation may be put in the following simpler forms:—

(1) $[x^2(x^3-m'^2yz)-y^2(m'z^2+xy)-z^2(m'y^2+zx)]^2=$
$$4\,(m'x^2+yz)^2\,(m'y^2+zx)\,(m'z^2+xy).$$

(2) $(x^6+y^6+z^6)-2\,(1+2m'^3)\,(x^3y^3+y^3z^3+z^3x^3)-$
$$6m'^2\,(x^3+y^3+z^3)\,xyz-3m'\,(m'^3+4)\,x^2y^2z^2=0$$

(3) $(x^3+y^3+z^3-3m'^2xyz)^2=4\,(1+m'^3)\,(x^3y^3+y^3z^3+z^3x^3+3m'x^2y^2z^2)$

The equation of the polar reciprocal of the Cayleyan with reference to the conic $x^2+y^2+z^2=0$ as derived from its tangential equation is $x^3+y^3+z^3+3m'xyz=0$; if x', y', z' be a point on this curve, its polar is $xx'+yy'+zz'=0$. The Cayleyan is the envelope of this line subject to the condition $x'^3+y'^3+z'^3+3m'x'y'z'=0$. In finding this envelope we arrive at form (1).

Form (1) of the equations above contains the extraneous factor x^2.

2. *The four points of contact of the lines forming the polar conics of a pair of conjugate poles with the Cayleyan are collinear.*

Every tangent to the Cayleyan cuts the Hessian of the given cubic in three points two of which form a pair of conjugate poles, and the third is the pole conjugate to that point on the Hessian at which the tangents to it at the conjugate poles meet; so AB joining the conjugate poles A and B cuts the Hessian again at C, which is the pole conjugate to D, the intersection of the tangents at them. AB and the lines forming the polar conic of B are the three tangents to the Cayleyan from A; similarly, AB and the lines forming the polar conic of A are the three tangents to the Cayleyan from B.

Let K be a point on the polar reciprocal of the Cayleyan: KX, KY, KZ, KW, are the four tangents to that cubic from K; the polar of K with respect to the given conic is a tangent to the Cayleyan and the pole of the tangent at K is its point of contact. Since four tangents to the polar reciprocal meet at K on it, by reciprocation the polar of K, a tangent to the Cayleyan, must pass through the four points of contact of the four tangents to the Cayleyan which are the polars of X, Y, Z, W. XY and WZ are a pair of opposite sides of the quadrilateral XYZW, and they must meet in a point L on the polar reciprocal. The tangents at K and L to the polar reciprocal meet at M on it, and KL cuts it in the third point R.

The three points X, Y, L lie on a right line; hence the three tangents to the Cayleyan which are the polars of these points must meet in a point A, the pole of that line; similarly the three tangents to the Cayleyan which are the polars of W, Z, L must meet in a point B which is the pole of the line WL; again, A and B lie on the polar of L which is a tangent to the Cayleyan. Thus AB is a tangent to the Cayleyan both from A and B, and they are a pair of conjugate poles on the Hessian of the given cubic; we know that the pair of right lines forming the polar conic of a conjugate pole touch the Cayleyan; so the polars of X and Y meeting at A form the polar

conic of the conjugate pole B: and the polars of Z and W form the polar conic of A. Thus the polar of the point K which is tangent to the Cayleyan passes through the points of contact with it of the four right lines forming the polar conics of the conjugate poles A and B i.e. the four points of contact are collinear.

We may extend this chain of arguments to find the conjugate poles C and D.

ANNOUNCEMENTS OF RESULTS AND CONJECTURES

Numbers representable by a Ternary Quadratic Form

I

1. It is well-known that every positive integer $\neq 4^a\,(8k+7)$ can be represented in the form: $x^2+y^2+z^2$, where x, y, z are integers. The general problem as to what numbers can be represented by the form: $ax^2+by^2+cz^2$ (where a, b, c are given positive integers) is still unsolved. However, the following special results are known:

TABLE[1]

Form	All numbers not representable by the form on the left are:
$x^2+y^2+z^2$	$4^a(8k+7)$
$x^2+y^2+4z^2$	$4^a(8k+7)$ or $8k+3$.
$x^2+y^2+5z^2$	$4^a(8k+3)$
$x^2+y^2+6z^2$	$9^a(9k+3)$
$x^2+y^2+8z^2$	$4^a(16k+14)$, $16k+6$, or $4k+3$.
$x^2+2y^2+6z^2$	$4^a(8k+5)$
$x^2+2y^2+8z^2$	$4^a(8k+7)$ or $8k+5$

(The table means that all numbers not representable by the form on the left are of the form on the right.)

2. At the suggestion of Dr. Chowla, I have attempted to solve the problem of numbers representable by the form: $x^2+y^2+11z^2$. My calculations have led me to the following:

Conjecture. *Odd numbers prime to* 11 *and* $\not\equiv 3$ (mod. 4) *are of the form* $x^2+y^2+11z^2$.

I have verified this conjecture for numbers up to 5000.

Other results in this direction are:

(i) *All even numbers which are not of the form:* $x^2+y^2+10z^2$ *are the numbers* $4^a(16k+6)$.

(ii) *All odd numbers* $>$ 2719 *are of the form:* $x^2+y^2+10z^2$. (H. Gupta's Conjecture[2].)

References

1. These results are extracted from Ramanujan's "*Collected Papers*", Cambridge, 1927, p. 172.

2. Some Idiosyncratic Numbers of Ramanujan. *Proc. Indian Acad. Sci.*, 13, 1941, p. 519.

Rohtak. SHER SINGH KADIAN.

II

We have undertaken to examine the types of numbers representable by the form $x^2+y^2+13z^2$. Calculations are still in progress. These calculations indicate at least one conjecture which is worth mentioning:

Conjecture:— *Every positive integer* $\equiv 1 \pmod{52}$ *is expressible in the form* $x^2+y^2+13z^2$ *where* x, y, z *are integers.*

This has been verified for every positive integer $\equiv 1 \pmod{52}$ and not exceeding 15601, e g. $(187.52)+1=24^2+19^2+13.26^2$.

It appears that the theory of ternary quadratic forms, as developed so far, is not powerful enough to prove results like the above conjecture. This theory is, however, perfectly capable of proving the results mentioned in our table, see Dickson, *Bulletin of the American Mathematical Society* Vol. 33, 1927, pp. 63–70. All attempts at *proving* results like our conjecture above, have so far been unsuccessful.

S. CHOWLA (*Lahore*) and
ABDUR RAHMAN NAZIR (*Quadian*)

III

Some New Congruence Properties of Ramanujan's Function $\tau(n)$

Defining $\tau(n)$ and $\sigma_k(n)$ by the relations

$$\sum_{n=1}^{\infty} \tau(n)\, x^n = x \prod_{n=1}^{\infty} (1 - x^n)^{24},$$

and

$$\sigma_k(n) = \sum_{d/n} d^k \; ; \; \sigma(n) = \sigma_1(n)$$

We have recently obtained the results

(1) $\tau(n) \equiv n^3\, \sigma(n) \pmod{2^4}$

(2) $\tau(n) \equiv \sigma_3(n) \pmod{2^5}$ if $(n, 2) = 1$

(3) $\tau(n) \equiv 2^{3m}\, \sigma_3(k) \pmod{2^5}$ if $n = 2^m \cdot k$; $(k, 2) = 1$; $m \geqslant 0$.

(4) $\tau(n) \equiv n^2\, \sigma_7(n) \pmod{3^3}$

(5) $\tau(n) \equiv k\, \sigma_7(n) \pmod{3^4}$ where $k = n^2$ if $n \equiv 0, 1 \pmod 3$ and $k = n^2 + 9$ if $n \equiv 2 \pmod 3$

(6) $\tau(3n) \equiv (3n)^2\, \sigma_7(3n) \pmod{3^5}$

(7) $\tau(n) \equiv n\, \sigma_9(n) \pmod{5^2}$

(8) $$\tau(n) \equiv 101\, n\, \sigma_9(n) + 5^2 \left\{ n\, \sigma_3(n) + (n-1)\, \sigma(n) - \frac{n\, \sigma_9(n) - n^3\, \sigma_7(n)}{5} \right\} \pmod{5^3}$$ if $(n, 5) = 1$

* (9) $\tau(n) \equiv n\, \sigma_3(n) \pmod 7$

(10) $$\tau(n) \equiv 8n^4\, \sigma_3(n) - 14 \left\{ 2(1 - n - n^3)\, \sigma_3(n) + (2n^2 - 3)\, \sigma(n) + \sum_{\substack{u + 7v = n \\ u, v \geqslant 1}} \sigma(u)\, \sigma_3(7v) \right\} \pmod{7^2}$$ if $(n, 7) = 1$.

---

† We conjecture $\tau(n) \equiv \sigma_{11}(n) \pmod{2^8}$ if $(n, 2) = 1$.

* After the despatch of our papers giving the result $\tau(n) \equiv n\, \sigma_3(n) \pmod 7$ to the Q. J. M. Oxford and the B. A. M. S. America, we have received the June 1945 issue of the J. I. M. S. which contains a paper of Ramanathan giving the same result. But two of the three proofs given by us are entirely different from Ramanathan's proof.

We also obtained new proofs of recent results of Gupta and Lahiri namely:

(11) $\tau(n) \equiv \sigma(n) \pmod{3}$ if $(n, 3) = 1$

(12) $\tau(n) \equiv \sigma(n) \pmod{8}$ if $(n, 2) = 1$.

Previous results in this direction are:

(13) $\tau(n) \equiv \sigma_{11}(n) \pmod{691}$

(14) $\tau(n) \equiv n\sigma(n) \pmod{5}$

(15) $\tau(jn-1) \equiv \sigma(jn-1) \pmod{j}$ if $j/24$.

(16) $\tau(23m+k) \equiv 0 \pmod{23}$ if $\left(\frac{k}{23}\right) = -1$.

(17) $\tau(7m+k) \equiv 0 \pmod{7}$ if $k=0, 3, 5, 6$.

(18) $\tau(3m+2) \equiv 0 \pmod{2}$

(19) $\tau(3^\lambda \cdot m) \equiv 0 \pmod{9^\lambda}$

(20) $\tau(2^\lambda \cdot m) \equiv 0 \pmod{8^\lambda}$

(21) $\tau(5^\lambda \cdot m) \equiv 0 \pmod{5^\lambda}$

(22) $\tau(7^\lambda \cdot m) \equiv 0 \pmod{7^\lambda}$.

These results are due to Ramanujan, Hardy, Watson, Mordell, Gupta, Wilton, Bannerji, and K. G. Ramanathan.

Combining these results with ours we can exhibit in a compact tabular forms as:

TABLE I

For all $n \geqslant 1$

1. $\tau(n) \equiv n\,\sigma(n) \pmod{2}$
2. $\tau(n) \equiv n\,\sigma(n) \pmod{3}$
3. $\tau(n) \equiv n^2\sigma(n) \pmod{4}$
4. $\tau(n) \equiv n\,\sigma(n) \pmod{5}$
5. $\tau(n) \equiv n\,\sigma(n) \pmod{6}$
6. $\tau(n) \equiv n\,\sigma_3(n) \pmod{7}$
7. $\tau(n) \equiv n^3\sigma(n) \pmod{8}$
8. $\tau(n) \equiv n^2\sigma(n) \pmod{9}$
9. $\tau(n) \equiv n\,\sigma(n) \pmod{10}$
10. $\tau(n) \equiv \sigma_{11}(n) \pmod{691}$
11. $\tau(23t+k) \equiv 0 \pmod{23}$ if $\left(\frac{k}{23}\right) = -1$.

TABLE II

1. $\tau(n) \equiv n\,\sigma_9(n) \pmod{5^2}$
2. $\tau(n) \equiv n^3\sigma(n) \pmod{2^4}$
3. $\tau(n) \equiv \sigma_3(n) \pmod{2^5}$ if $(n, 2)=1$
4. $\tau(2^\lambda n) \equiv 0 \pmod{2^{3\lambda}}$
5. $\tau(3^\lambda n) \equiv 0 \pmod{3^{2\lambda}}$
6. $\tau(n) \equiv 2^{3m}\,\sigma_3(k) \pmod{2^5}$ when $n=2^m\cdot k$, $m \geqslant 1$, k odd.

TABLE III

1. $\tau(n) \equiv n\,\sigma(n) \pmod{j}$ where $j/6$
2. $\tau(n) \equiv n^2\sigma(n) \pmod{j}$ where $j/12$
3. $\tau(n) \equiv n^3\sigma(n) \pmod{j}$ where $j/48$
4. $\tau(n) \equiv \sigma_3(n) \pmod{j}$ where $(n, j)=1$ and $j/96$
5. $\tau(n) \equiv n^8\sigma(n) \pmod{j}$ where $j/72$
6. $\tau(n) \equiv n\,\sigma(n) \pmod{j}$ where $j/30$
7. $\tau(n) \equiv n\,\sigma_3(n) \pmod{j}$ where $j/42$
8. $\tau(n) \equiv \sigma_{11}(n) \pmod{j\ 691}$ where $j/96$ and $(n, j) = 1$.

§ 2. From these tables it is easy to verify Walfisz' Theorem viz.

$$\tau(n) \equiv 0 \pmod{2^5.3^2.5^2.7.691}$$

for almost all n, because

$$\sigma_k(n) \equiv 0 \pmod{s}$$

for all s, all odd k and almost all n.

§ 3. S. Chowla has also observed that

$$\tau(n) \equiv 0 \pmod{23}$$

for almost all n.

So that we can extend Walfisz' Theorem to

$$\tau(n) \equiv 0 \pmod{2^5.3^2.5^2.7.23.691}$$

R. P. BAMBAH,
Research Student, Lahore.

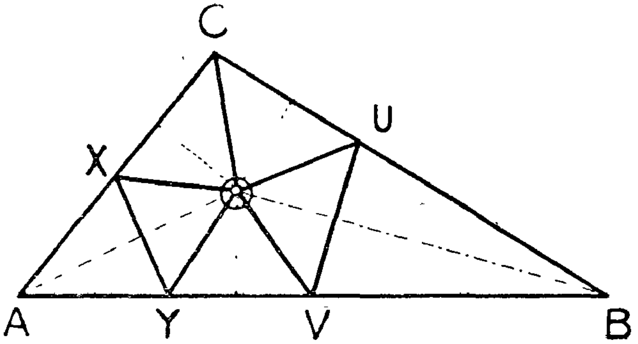
S. CHOWLA,
Govt. College, Lahore.

NOTES AND DISCUSSIONS

Triangulation with acute-angled triangles

The problem offered itself to the author many years ago, when perusing a book[1] on Calculus, where the theory of the area on a curved surface is based on the triangulation of its plane projection into acute-angled triangles. The existence of such a triangulation was not discussed.

As every plane domain which is bounded by straight segments, can be triangulated[2], it suffices to show that every triangle with a right or an obtuse angle can be dissected into acute-angled triangles. A construction which gives always the required result, is given by the diagram inserted here. O is the centre of the inner circle of the



triangle ABC which has at C a right or obtuse angle. The points X, Y, U, V are obtained by bisecting the angles between OA (resp. OB) and the perpendiculars from O on the sides of the triangle. An easy calculation (left to the reader) shows that the 21 angles concerned are necessarily acute. A dissection of an obtuse-(or right-) angled triangle into less than seven acute-angled triangles is not possible.

F. W. Levi

---

[1] In the later editions, a different method has been applied; the original edition is not available to me at present.

[2] See: B. V. Kérekjartó; Vorlesungen über Topologie (Berlin, 1923), p. 21–23. F. Levi; Geometrische Konfigurationen (Leipzig 1929), p. 84.

Residual Types of Partitions of "0" into Four Cubes

1. In a paper* under the above title, A. K. Srinivasan examines the residual types of four integers a, b, c, d such that

$$a^3+b^3+c^3+d^3=0,$$

and reduces the problem to ten types of partitions. Finding no examples available for his types III, IV and IX, he "feels that they are probably non-existent." The object of this note is to prove that these types actually do not exist.

2. We have, for any integer m,

$$\begin{aligned}(6m)^3 &\equiv 0 \pmod{9}\\ (6m+1)^3 &\equiv 1 \pmod{9}\\ (6m+2)^3 &\equiv 8 \pmod{9}\\ (6m+3)^3 &\equiv 0 \pmod{9}\\ (6m+4)^3 &\equiv 1 \pmod{9}\\ (6m+5)^3 &\equiv 8 \pmod{9}\end{aligned} \qquad \cdots\cdots (1)$$

If type III, viz. (01^24), exist, it would imply the existence of integers p, q, r, s such that

$$(6p)^3+(6q+1)^3+(6r+1)^3+(6s+4)^3=0$$

But from (1), this implies $0+1+1+1\equiv 0 \pmod{9}$ which is false. Hence type III does not exist.

If type IV, viz. (1^33), exist, the existence of integers p, q, r, s would be implied such that

$$(6p+1)^3+(6q+1)^3+(6r+1)^3+(6s+3)^3=0$$

But by (1), taking residues (mod. 9), we shall have $1+1+1+0\equiv 0 \pmod{9}$. Hence type IV does not exist.

If type IX, viz. (2^235), exist, we must have

$$(6p+2)^3+(6q+2)^3+(6r+3)^3+(6s+5)^3=0$$

which implies $8+8+0+8\equiv 0 \pmod{9}$ which is false.

$\therefore$ type IX does not exist.

It is, therefore, no wonder that Russel and Gwythers' † "*Tables of Partition*" contain no illustrations of these types.

Presidency College, Calcutta.

S. M. KERAWALA

* *Math. Student*, Vol. XIII, 2, pp. 47-48.
† *Math. Gaz.* Vol. XXI, pp. 34-35.

A note on the Equation $a^m x + b^n y = 1$.

The object of this note is to show that a solution in integers of the equation

$$a^m x + b^n y = 1 \qquad \dots \quad (1)$$

can be written down as soon as we know a solution of

$$ax + by = 1 \qquad \dots \quad (2)$$

Theorem If $x = \xi$, $y = \eta$ is a solution of (2), then a solution of (1) is given by

$$x = \xi^m \, P_{m,n}(b\eta), \quad y = \eta^n \, P_{n,m}(a\xi)$$

where $P_{m,n}(t)$ denotes the sum of the first n terms in the formal expansion of $(1-t)^{-m}$ in ascending powers of t.

This theorem is an immediate consequence of the following *lemma*:

If $x + y = 1$, then

$$x^m \, P_{m,n}(y) + y^n \, P_{n,m}(x) = 1.$$

We shall give two different proofs of this lemma.

Firstly, let $x^m \, P_{m,n}(y) + y^n \, P_{n,m}(x)$ be denoted by $Q_{m,n}$ Then

$$Q_{m+1,n} = x^{m+1} \, P_{m+1,n}(y) + y^n \, P_{n,m+1}(x)$$

$$= x^m(1-y) \left\{ 1 + \binom{m+1}{1} y + \binom{m+2}{2} y^2 + \cdots + \binom{m+n-1}{n-1} y^{n-1} \right\}$$
$$+ y^n \left\{ P_{n,m}(x) + \binom{m+n-1}{m} x^m \right\}$$

$$= x^m \left\{ 1 + \binom{m}{1} y + \binom{m+1}{2} y^2 + \cdots + \binom{m+n-2}{n-1} y^{n-1} - \binom{m+n-1}{n-1} y^n \right\}$$
$$+ y^n \left\{ P_{n,m}(x) + \binom{m+n-1}{n-1} x^m \right\}$$

$$= x^m \left\{ P_{m,n}(y) - \binom{m+n-1}{n-1} y^n \right\} + y^n \left\{ P_{n,m}(x) + \binom{m+n-1}{n-1} x^m \right\}$$

$$= Q_{m,n}$$

Similarly $Q_{m,n+1} = Q_{m,n}$.

Therefore $Q_{m,n} = Q_{m,n-1} = Q_{m,n-2} = \cdots = Q_{m,1}$
$= Q_{m-1,1} = Q_{m-2,1} = \cdots = Q_{1,1}$

But $Q_{1,1} = x + y = 1$. Hence $Q_{m,n} = 1$.

Secondly we note that x^m and $y^n = (1-x)^n$ are mutually prime polynomials in x. Hence there exist two polynomials X, Y of degrees less than n and m respectively in x such that

$$x^m X + y^n Y = 1$$

Now $\frac{1}{y^n} = P_{n,m}(x) + x^m P.$

where P is a formal power series in x.

Therefore $1 = y^n P_{n,m}(x) + x^m(y^n P)$.

But $y^n P_{n,m}(x)$ is a polynomial in x of degree $n+m-1$. It follows that $y^n P$ must also be a polynomial in x of degree $n-1$.

Hence $X = y^n P$, $Y = P_{n,m}(x)$.

Considerations of symmetry give $Py^n = P_{m,n}(y)$ whence the required result follows.

Replacing x by ax and y by by in the lemma we get that

if $ax+by=1$, then $a^m\{x^m P_{m,n}(by)\} + b^n\{y^n P_{n,m}(ax)\} = 1$,

which proves the theorem.

Christian College,
Tambaram.

P. KESAVA MENON

Extension of Pascals' Theorem

The theorem that the pairs of opposite sides of a hexagon inscribed in a conic meet in collinear points may be extended as follows:—

"If a polygon of $2n$ sides ($n \geqslant 3$) be inscribed in a conic and each of the n sides belonging to one system of alternate sides intersect the $(n-2)$ sides, other than the adjacent ones, belonging to the other system in $n-2$ points, the $n(n-2)$ points so obtained lie on a curve of degree $n-2$"

The converse is also true.

The proof, based on properties of n-ics, is quite simple. In general $\frac{n(n+3)}{2}$ points define a unique n-ic so that there are an infinity of n-ics through $\frac{n(n+3)}{2} - 1$ given points. If two of the n-ics are $S=0$ $S'=0$, the rest are given by $S + \lambda S' = 0$, λ being an arbitrary parameter. It follows at once that all of them pass through the n^2 points of intersection of S and S′. Thus we see that n-ics through $\frac{n(n+3)}{2} - 1$ given points pass also through $\frac{(n-1)(n-2)}{2}$ other fixed points.

Suppose $2n$ of the given $\frac{n(n+3)}{2} - 1$ points lie on a conic. The remaining points are $\frac{(n-2)(n+1)}{2}$ in number and hence a curve of degree $n-2$ passes through them. The conic and the $(n-2)$-ic together form an n-ic passing through the given points. It follows that the $(n-2)$-ic will also pass through the $\frac{(n-1)(n-2)}{2}$ fixed points defined by the given points provided none of them lies on the conic.

Now the two systems of n alternate sides of the $2n$-gon inscribed in a conic form two n-ics intersecting in $n^2 - 2n$ other points. If, therefore, an $(n-2)$-ic is made to pass through $\frac{(n-1)(n+2)}{2}$ of the latter, it will, according to the above arguments pass also through the remaining $\frac{(n-1)(n-2)}{2}$ points of intersection since none of the latter can be on the conic. (Each of the $2n$ sides of the polygon already intersects the conic in two of the vertices of the polygon.)

Corollary 1. When $n=4$ the theorem may be stated as follows:

If an octagon be inscribed in a conic and each side intersects the fourth from itself (reckoned in a definite sense,) in a point, the eight points so obtained lie on a conic.

Corollary 2 If A, B, C, D; A′, B′, C′, D′, be two tetrads of points on two intersecting lines, the eight points of intersection of the pairs of cross-joints AA′, B′C; BB′, C′D; CC′, D′A; DD′, A′B; A′B, CC′; B′C, DD′; C′D, AA′; D′A, BB′ lie on a conic.

Christian College,
Tambaram.

P. KESAVA MENON.

[Vide in this connection a paper by W. K. Clifford on "Analogues of Pascal's Theorem". Collected Mathematical Papers, 1882, pp. 72–79.

—EDITOR.]

COLLEGIATE SECTION

Riemann Integral as the limit of a product

Let $0 < a < b$, and let $f(x)$ be an integrable function in the region (a, b).

If $\quad I = \prod_{1}^{n} (x_p/x_{p-1})^{f(x_{p-1})}$

then $\log I = \sum_{1}^{n} f(x_{p-1}) \log (x_p/x_{p-1})$

$$= \sum f(x_{p-1}) \left\{ \frac{x_p - x_{p-1}}{x_{p-1}} + O(\eta_p^2) \right\}.$$

where $\eta_p = x_p - x_{p-1}$ since $x_p > a > 0$.

Hence $\lim \log I = \int_a^b \frac{f(x)}{x} dx$ since $\sum | f(x_{p-1}) \eta_p^2 | \to 0$ by the remark above. Hence I tends to the limit $\exp. \int_a^b \frac{f(x)}{x} dx$.

Trichur. GEORGE A. PANAGAT.

Solution of the cubic equation by inspection

1. The following method is an extension to cubic equations of the familiar rule for solving the quadratic $x^2 + Bx + C$ by expressing B as the sum of two numbers whose product is C.

Let $Ax^3 + Bx^2 + Cx + D$ be a cubic expression, and let numbers p, q, p', q' be found such that

$$\text{(i)} \quad B = p + q, \qquad C = p' + q'$$
$$\text{(ii)} \quad pq/A = p', \quad p'q'/D = p$$

then $Ax + q$ is a factor of the cubic.

Proof:

From (ii) we have also $\frac{q}{A} = \frac{D}{q'}$.

Now $Ax^3 + Bx^2 + Cx + D = Ax^3 + (p+q)x^2 + (p' + q')x + D$

$$= x^2(Ax + q) + x(px + p') + (q'x + D)$$
$$= x^2(Ax + q) + x\left(\frac{Ap'}{q} x + p'\right) + \left(\frac{AD}{q} x + D\right)$$
$$= (Ax + q)\left(x^2 + \frac{p'}{q'} x + \frac{D}{q}\right)$$ which proves the result.

14 DEC 1944 MADRAS

2. Thus to solve $x^3+6x^2+11x+6=0$ we note that $C=11$ and $11=p'+q'=6+5$ with $p'q'\div D=5$. Again $B=6$ is broken into $p+q$ with $p=5$ and so $q=1$ with $pq/A=5=q'$. Hence $Ax+q=x+1$ is a factor. On dividing by $x+1$ we get a quadratic which can be further factored.

Example 2. Solve $28x^3-9x^2+1=0$.

Here $C=0$ and we write it as $4+(-4)=p'+q'$.

Also $p'q'/D= -16$ and this must be p or q.

So we write $B=-9=-16+7=p+q$ and $p\times q/A=-16\times 7/28=-4$ which is one of the parts into which C has been split. Hence $Ax+q=28x+7$ *i.e.* $4x+1$ is a factor. Dividing by it the remaining quadratic is $7x^2-4x+1$. Hence the roots are $-\frac{1}{4}, \dfrac{2\pm i\sqrt{3}}{7}$.

3. Like all such methods, success depends on correct guessing, and repeated trial with likely numbers. The method has the merit of dealing with equations where A is not unity, and thus having rational non integral roots.

S. D. College, Lahore. R. S. LUTHER, Student M. A. Class.

On finding a particular integral of a linear differential equation with constant co-efficients

The following "long division method" is offered as an alternative to the one given in Forsyth's book.

Let the differential equation be

$$\phi(D)y\equiv(a+bD+cD^2 \cdot\cdot\cdot\cdot\cdot)y=f(x) \qquad \cdots (1)$$

where D stands for $\dfrac{d}{dx}$. We consider first the case when $f(x)=cx^m$, and illustrate it by means of the following

Example 1.

$$2y-4\frac{dy}{dx}+5\frac{d^2y}{dx^2}=7x^3 \qquad \cdots (2)$$

writing D for $\dfrac{d}{dx}$, we have the following long division which is self explanatory.

$$2-4D+5D^2)7x^3 \qquad (\tfrac{7}{2}x^3+21x^2+\tfrac{63}{2}x-42$$

$$\underline{7x^3-42x^2+105x}$$

$$42x^2-105x$$

$$\underline{42x^2-168x+210}$$

$$63x-210$$

$$\underline{63x-126}$$

$$-84$$

$$\underline{-84}$$

So a particular integral is $\frac{7}{2}x^3 + 21x^2 + \frac{63}{2}x - 42$. The above method is applicable even if a or a, b etc. are zero.

If $f(x)$ is of the form $x^m e^{nx} \cos qx$ (or $\sin qx$) we may reduce this to the case of the above example thus. Suppose $a, b, c \cdots$ in equation (1) are real. Then a particular integral of (1) is the real or imaginary part of u where $\phi(D)\, u = x^m e^{(n+iq)x}$, $D \equiv \frac{d}{dx}$. Putting $u = w\, e^{(n+iq)x}$ we have $\phi(D)\, \{e^{(n+iq)x}\, w\} = e^{(n+iq)x}\, \phi(D+n+iq)\, w$ so that w satisfies the equation $\phi(D+n+iq)w = px^m$ which is of the type of example 1.

Vizagapatam. S PURUSHOTHAM.

Partial Fractions

1. *Notation*: In what follows, 'pf' means partial fraction; 'pf : $A(x)$' means partial fraction corresponding to the factor $A(x)$ in the denominator of a given fraction; '$R\left[\frac{A(x)}{B(x)}\right]$' means the remainder left when $A(x)$ is divided by $B(x)$; '$R\left[\frac{A(x)}{\{g(x)\}^k}\right]$' means the remainder expressed as $A_0 + A_1 g(x) + \cdots + A_{k-1}\{g(x)\}^{k-1}$ *where each A is lower in degree than* $g(x)$; "$\mp$" denotes equivalence under a stated condition as will be clear from the sequel.

2. *Introduction*: In breaking a given fraction into pfs. we have the well-known rule for writing down the pf. corresponding to a non-repeated linear factor of the denominator. For example, pf : $(x-a)$ in $\frac{N(x)}{(x-a)\,H(x)} = \frac{N(a)}{(x-a)\,H(a)}$.

In this paper we obtain a similar rule which holds for *any* factor, whatever its degree and multiplicity, provoided only that it is prime to every other factor of the denominator. It will be seen that the rule avoids the setting up of unknowns, as is usually done.

3. *Theorem*: If, G, H, M, N be any polynomials in x of which G is prime to each of H, M, then pf : G is the same in $\frac{N}{GH}$ and $\frac{R(NM/G)}{G \cdot R(HM/G)}$

In other words, for pf : G,

$$\frac{1}{G}\frac{N}{H} \mp \frac{1}{G} \cdot \frac{R(NM/G)}{R(HM/G)} \qquad \cdots\cdots (3{\cdot}1)$$

Proof:

$$\text{Let } \frac{N}{GH} = I + \frac{A}{G} + \frac{B}{H}.$$

Also let

$$R\left(\frac{NM}{G}\right) = \rho;\ R\left(\frac{HM}{G}\right) = \rho';\ NM = \lambda G + \rho;\ HM = \mu G + \rho':$$

$$\frac{\rho}{G\rho'} = \frac{C}{G} + \frac{D}{\rho'} \qquad \cdots\cdots (3\cdot 2)$$

where I, A, B, ρ, ρ', λ, μ, C, D are polynomials in x; A, C, ρ, ρ' are lower in degree than G; and B, D are lower in degree than H, ρ' respectively. Then, from (3·2),

$$\rho = \rho' C + GD.$$

i. e. $\quad NM - \lambda G = (HM - \mu G) C + GD \qquad \cdots\cdots (3\cdot 3)$

But, $\quad N = IGH + AH + BG$

$\therefore$ from (3·3) $\quad HM(C - A) = G(IHM + BM + C\mu - D - \lambda) \quad \cdots (3\cdot 4)$

But G is prime to HM. So, C = A, since, C, A are each lower in degree than G.

4. *Remark*: In applying the theorem, if M be such that $R\left(\frac{HM}{G}\right)$ is a constant, then $A = R\left(\frac{NM}{G}\right)/R\left(\frac{HM}{G}\right)$. Such M is sometimes easily got by inspection.

In the general case, M may be found by the lemma. "Given two polynomials in x, say G, H which are prime to one another, there exists a *polynomial* in G (with constant co-efficients) which is prime to G but is divisible by H."

Proof:

Let H be of degree n in x. Divide G, G^2, $\cdots$ G^n by H and let

$$G = aH + p_1x^{n-1} + p_2x^{n-2} + \cdots\cdots + p_{n-1}x + p_n$$
$$G^2 = bH + q_1x^{n-1} + \cdots\cdots\cdots + q_{n-1}x + q_n$$
$$G^n = cH + r_1x^{n-1} + \cdots\cdots\cdots + r_{n-1}x + r_n$$

Eliminating x, x^2, $\cdots$ x^{n-1} from the above, we have

$$H\begin{vmatrix} a & p_1 & \cdot & \cdot & p_{n-1} \\ b & q_1 & \cdot & \cdot & q_{n-1} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ c & r_1 & \cdot & \cdot & r_{n-1} \end{vmatrix} = \begin{vmatrix} G - p_n & p_1 & \cdot & \cdot & p_{n-1} \\ G^2 - q_n & q_1 & \cdot & \cdot & q_{n-1} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ G^n - r_n & r_1 & \cdot & \cdot & r_{n-1} \end{vmatrix}$$

or briefly $HM' = P(G)$ a polynomial in G with constant co-efficients. If P(G) is of the form $G^k\,Q(G)$ where Q(G) is not divisible by G, we have $\frac{HM'}{G^k} = Q(G)$. Since G is prime to H, M′ is divisible by

G^k the quotient being say M. Then $HM=Q(G)$ is the required polynomial. Such an M will make $R(HM/G)$ a constant, so that theorem (3·1) becomes easy to apply.

5. When $G=g^k$, where g is a polynomial in x, the above method, or the following rule, whose proof is left to the reader, will enable the partial quotients corresponding to $g^{k'}$ to be calculated.

Let $\frac{N}{g^k H}=I+\frac{A_1}{g}+\frac{A_2}{g^2}\cdots+\frac{A_k}{g^k}+\frac{B}{H}=\frac{A}{g^k}+\frac{B}{H}$ say

Then $A=R\left\{\frac{(NM)(HM)^{-1}}{g^k}\right\}$ where M is prime to g such that HM is a polynomial $Q(g)$ in g with constant co-efficients, $(HM)^{-1}$ is the formal expansion of $\frac{1}{Q(g)}$ as a power series in ascending powers of g, and NM is also expressed as a polynomial in g whose co-efficients are of a lower degree than g.

Rajahmundry. K. SATYANARAYANA.

Test for divisibility by numbers ending in 9

The usual test for divisibility by 9 is based on successive applications of the congruence $10a+b\equiv a+b \pmod 9$. A generalization is obtained by using the following result:

$10a+b$ is or is not divisible by $10r+9$ according as $a+(r+1)b$ is or is not divisible by $10r+9$. .... (A)

Assuming the truth of the above, we may reduce the test of the divisibility of say 21302 by 19 to that of $2130+4=2134$ by 19, and this again to that of $213+8=221$ and this to $22+2=24$. Hence 21302 is not divisible by 19.

To prove the result A, let it be assumed that

$$10a+b\equiv c \bmod (10r+9),\ c<(10r+9).$$

Then $$10a(r+1)+b(r+1)\equiv c(r+1) \bmod (10r+9)$$

That is $$a+b(r+1)\equiv c(r+1) \because 10ra\equiv -9a$$

Hence if $c\equiv 0 \bmod (10r+9)$ so also is $a+b(r+1)$. If conversely $a+b(r+1)\equiv 0 \bmod (10r+9)$, so is $c(r+1)$. Now any common factor of $r+1$ and $10r+9$ will also be a factor of $10r+9-9(r+1)=r$ and hence of 1. As $r+1$ is prime to $(10r+9)$, it follows that c is divisible by $10r+9$, that is c is 0.

If a number a' can be found such that $aa' \equiv 1$, then obviously $c \equiv a' + b$. The above test has the unusual feature that it does not replace the original number by a simpler one congruent with it, for our result (A) does not assert that $10a + b$ and $a + (r+1)b$ are congruent mod $(10r + 9)$.

University College,
Trivandrum.

K. BHUTALINGAM,
Student, Senior B.Sc.

On testing for divisibility by 7

Let $N = a_1 + 10a_2 + 10^2 a_3 + \cdots, \; 0 \leqslant a_1, a_2 \cdots < 9$.

Also let the remainders when $1, 10, 10^2, \cdots$ are divided by a divisor d be $r_1, r_2, r_3, \cdots$ etc. Then obviously the remainder R when N is divided by d is

$$R \equiv r_1 a_1 + r_2 a_2 + \cdots\cdots \quad (\text{mod. } d)$$

Thus for $d = 7$ the numbers $r_1 \, r_2 \cdots$ are 1, 3, 2, 6, 4, 5 after which they repeat. Writing these as $1, 3, 2, -1, -3, -2$ (mod 7) we see that the remainder when N is divided by 7 is

$$(a_1 + 3a_2 + 2a_3) - (a_4 + 3a_5 + 2a_6) + \cdots$$
$$\cdots + (-1)^{n-1}(a_{3n-2} + 3a_{3n-1} + 2a_{3n})$$
$$= P_1 + P_2 + \cdots + P_n$$

Where $P_n = (-1)^{n-1}(a_{3n-2} + 3a_{3n-1} + 2a_{3n})$

Now let x_{n-1} be a positive integer less than 7 such that $|P_n| + x_{n-1}$ is a multiple of 7. Two cases arise according as P_n is positive or negative. If P_n is positive, P_{n-1} is negative and $P_n + x_{n-1} = 7k$ where k is a positive integer.

$$\therefore \quad P_n + P_{n-1} = P_n + x_{n-1} - x_{n-1} - |P_{n-1}| = 7k - (x_{n-1} + |P_{n-1}|)$$

If P_n is negative, P_{n-1} is positive and $P_n - x_{n-1} = -7k$

$$\therefore \quad P_n + P_{n-1} = P_n - x_{n-1} + x_{n-1} + |P_{n-1}| = -7k + (x_{n-1} + |F_{n-1}|)$$

In either case we see that $x_{n-1} + |P_{n-1}|$ taken with the sign of P_{n-1} represents a number differing from $P_n + P_{n-1}$ by a multiple of 7. Hence we may write

$$|E_{n-1}| = x_{n-1} + |P_{n-1}|$$

where E_{n-1} has the same sign as P_{n-1} and is equivalent to $P_n + P_{n-1}$. Now we have to evaluate

$$E_{n-1} + P_{n-2} + P_{n-3} + \cdots + P_1$$

As before let x_{n-2} be a positive integer less than 7 such that $|E_{n-1}| + x_{n-2}$ is a multiple of 7. Then

$$|E_{n-2}| = x_{n-2} + |P_{n-2}|$$

where E_{n-2} has the same sign as P_{n-2} and is equivalent to $P_n+P_{n-1}+P_{n-2}$. Obviously by continuing the process we finally arrive at a number E_1 given by $|E_1|=x_1+|P_1|$ where E_1 has the same sign as P_1 and is equivalent to the remainder

$$P_n+P_{n-1}+\cdots\cdots+P_1$$

As P_1 is positive E_1 is likewise positive. It is to be noted that in calculating E_1 only the numerical values of P_1 P_2 $\cdots$ etc. are used, and so we get the following rule:—

Divide the digits into sets of 3 beginning from the units digit and proceeding left. A sufficient number of zero digits must be inserted, if necessary, at the extreme left to make up a final group of 3 digits. To obtain $|P_n|$ run along the last group of 3 digits in the order in which they occur thus:—

$$a_{3n}+a_{3n-1}+a_{3n-2}+a_{3n-1}+a_{3n}+a_{3n-1}$$

going forwards and backwards and ending with the middle digit. Their sum is

$$|P_n|=a_{3n-2}+3a_{3n-1}+2a_{3n}$$

Now find the number $x_{n-1}<7$ which when added to $|P_n|$ gives a multiple of 7. Carry over x_{n-1} and add it to $|P_{n-1}|$ thus getting $|E_{n-1}|$. Repeat the same process until E_1 is obtained. The true remainder R resulting from the division of N by 7 is E_1 reduced (mod 7). From this it follows that N cannot be exactly divisible by 7 unless E_1 is a multiple of 7.

Example. Let $N=16,439,852$

$0+1+6+1+0+1=9$ carry over 5 since $9+5$ is a multiple of 7.

$5+4+3+9+3+4+3=31$ carry over 4

$4+8+5+2+5+8+5=37$

∴ The remainder when N is divided by 7 is 2.

Govt. Teleproph Office, Vizagapatam.

S. VENKATARAMAN.

The age of the moon on any day

Mr. Forbe's method (Vide *The Mathematics Student* June 1945, page 76) of calculating the age of the moon on any day in any year leads to an error of two days in same cases. The "Astronavigation for the Soldier" India, 1945 gives the following method for calculating the phase of the moon:—"Add the epact, date of the month plus number of the month. Each year the epact increases by 11

commencing again at 1 after reaching 30. The epact for 1945 is 16; for 1946 it is 27, for 1947, 8 and so on. The months are numbered commencing from March as the first month and February as the 12th: for January and Feb. the epact of the previous year must be used. The whole sum, epact + date + number of month should not exceed 30. If the sum is over 30, subtract 30 and the answer represents the age of the moon, zero being equivalent to new moon, 15 full moon." The error rarely exceeds one day.

Combining these two methods, a new one could be suggested:— Subtract 1920 from the year, multiply the remainder by 11, add month, add date, add also 9 in the case of all months March—Dec. and 8 in the case of January and February. Reduce *modulo* 30. The result is the age of the moon.

Karachi. K. S. B. SASTRI.

Some Instructive Problems and their solutions—II

[Under this heading we shall publish from time to time, problems of special interest to students preparing for examinations, together with solutions and comments on them.—*Editor*.

Analysis

I. Discuss the behaviour of the sequence (u_n) where $u_{n+1}=3-\dfrac{2}{u_n}$

If $u_1=1$, then it is easily seen that $u_n=1$ for all n. If $u_1=2$, then $u_n=2$ for all n. We shall now suppose that $u_n \neq 1$ or 2. The given relation can be written as

$$\frac{u_{n+1}-1}{u_{n+1}-2}=2\,\frac{u_n-1}{u_n-2}=2^n\,\frac{u_1-1}{u_1-2}.$$

Hence $\left|\dfrac{u_{n+1}-2}{u_{n+1}-1}\right| \to 0$ as $n \to \infty$. That is $\dfrac{1}{u_{n+1}-1} \to 1$ as $n \to \infty$ so that $u_{n+1} \to 2$ as $n \to \infty$. Hence in all cases, $u_n \to 1$ (in case $u_1=1$) or $u_n \to 2$ (in case $u_1 \neq 1$).

The above is a special case of the general problem of discussing the nature of the sequence (u_n) where

$$u_{n+1}=\frac{au_n+b}{cu_n+d},\quad ad-bc \neq 0 \qquad \cdots \quad (1)$$

The behaviour of u_n depends on the nature of the fixed points of the transformation

$$z' = \frac{az+b}{cz+d}$$

(Here no restriction is made on the numbers a, b, c, d. They may be real or complex). Let α, β be the roots of

$$ct^2+(d-a)t-b=0.$$

We can take

$$\alpha = \frac{1}{2c}\left\{a-d+\sqrt{[(a-d)^2+4bc]}\right\}, \quad \beta=\frac{1}{2c}\left\{a-d-\sqrt{[(a-d)^2+4bc]}\right\}$$

Case (i) Let $\alpha \neq \beta$, $c \neq 0$. Then (1) can be written as

$$\frac{u_{n+1}-\alpha}{u_{n+1}-\beta} = \lambda \cdot \frac{u_n-\alpha}{u_n-\beta} \qquad \ldots \quad (2)$$

where $\lambda = \dfrac{c\beta+d}{c\alpha+d}$. If $u_1=\alpha$ or β, then $u_n=\alpha$ or β for all n. If $u_1 \neq \alpha$ or β, then $u_n \to \alpha$ if $|\lambda|<1$ and $u_n \to \beta$ if $|\lambda|>1$. If $|\lambda|=1$, say $\lambda=e^{i\theta}$, θ real, then $\dfrac{u_{n+1}-\alpha}{u_{n+1}-\beta} = \dfrac{u_1-\alpha}{u_1-\beta}e^{n\theta}$. If $\theta=0$ then $u_n=u_1$. If $0<\theta<2\pi$, the behaviour of u_n will depend on the nature of the number $\dfrac{\theta}{\pi}$. If $\dfrac{\theta}{\pi}$ is rational, (u_n) will have a finite number of limit points. If $\dfrac{\theta}{\pi}$ is irrational then (u_n) will have a continuum of limit points.

Case (ii) Let $\alpha \neq \beta$, $c=0$. Here one of the fixed points $\alpha=\infty$. We get $u_{n+1}-\beta = \dfrac{a}{d}(u_n-\beta)$ where $\beta = \dfrac{b}{d-a}$. As before if $\left|\dfrac{a}{d}\right|<1$, $u_n \to \beta$ as $n \to \infty$. If $\left|\dfrac{a}{d}\right|>1, |u_n| \to \infty$. If $\left|\dfrac{a}{d}\right|=1$, the behaviour of u_n will depend, as on θ where $\dfrac{a}{d}=e^{i\theta}$

Case (iii) $\alpha=\beta$, $c \neq 0$ we can write

$$\frac{1}{u_{n+1}-\alpha} = \frac{1}{u_n-\alpha} + \frac{2c}{a+d}$$

Here $\dfrac{1}{u_n-\alpha} \to \infty$ or $u_n \to \alpha$.

Case (iv) $\alpha=\beta$, $c=0$. Then $a=d$ and we get

$$u_{n+1} = u_n + \frac{b}{a}$$

So $u_n \to \infty$ in this case.

This gives a complete discussion of the sequence (u_n) satisfying (1).

II. Discuss the nature of the sequence (u_n) where

$$u_{n+1}=\sqrt{\left(\frac{u_n^2+ab^2}{a+1}\right)},\ a>0\ ;\ b^2>0.$$

We get the two relations

$$u_{n+1}^2-b^2=\frac{u_n^2-b^2}{a+1}$$

and

$$u_{n+1}^2-u_n^2=\frac{a(b^2-u_n^2)}{a+1}$$

Case (i) $u_1 \leqslant b$. Then the first of the above relation shows that $u_n \leqslant b$ for all n and the second one then shows that (u_n) is a monotonic-increasing sequence. Hence $u_n \to$ a limit λ and $\lambda^2=\dfrac{\lambda^2+ab^2}{a+1}$ or $\lambda=b$.

Case (ii) $u_1 \geqslant b$ Here we get $u_n \geqslant b$ for all n and so (u_n) is a decreasing sequence. Hence in this case also $u_n \to b$.

III. Let f(x) be defined and have a finite derivative for all x. Let the number θ $(0<\theta<1)$ in the equation

$$f(x+h)=f(x)+hf'(x+\theta h) \quad \cdots\cdots (1)$$

be independent of x and h. Then if $\theta \neq \frac{1}{2}$, prove that f(x) is a linear function and if $\theta=\frac{1}{2}$, f(x) is a quadric function.

Proof: Case (i) Let $\theta \neq \frac{1}{2}$ so that $0>p=1-\dfrac{1}{\theta} \neq -1$. Putting $x+\theta h=0$ in (1) we get

$$f(px)=f(x)+x(p-1)f'(0) \quad \cdots\cdots (2)$$

which is true for all values of x. Replacing x by px, p^2x, $\cdots p^{n-1}x$ successively in (2) and adding we get

$$f(p^nx)=f(x)+(p^n-1)xf'(0) \quad \cdots\cdots (3)$$

If $0>p>-1$, $p^n \to 0$ as $n\to\infty$. Hence (3) gives $f(0)=f(x)-xf'(0)$ i.e. $f(x)=f(0)+xf'(0)$ since $f(x)$ is continuous for all x. Hence $f(x)$ is linear in this case. If $p<-1$, put $p^nx=u$ so that (3) becomes

$$f(u)=f\left(\frac{u}{p^n}\right)+\left(1-\frac{1}{p^n}\right)uf'(0)$$

and letting $n\to\infty$ in this equation we get $f(u)=f(0)+uf'(0)$. So the result is true in this case also.

Case (ii) $\theta=\dfrac{1}{2}$ so that $p=-1$. Let $x+\dfrac{h}{2}=\lambda$. Then (1) gives

$$f(2\lambda-x)=f(x)+2(\lambda-x)f'(\lambda) \quad \cdots\cdots (4)$$

Treating λ as a constant and differentiating (4) with respect to x,

$$-f'(2\lambda-x)=f'(x)-2f'(\lambda) \quad \cdots\cdots (5)$$

Putting $\lambda=0$ in (5) and subtracting we get

$$f'(-x)-f'(2\lambda-x)=2[f'(0)-f'(\lambda)] \quad \cdots\cdots (6)$$

Replacing x by $-x$ and rewriting, (6) gives

$$f'(2\lambda+x)-2f'(\lambda)=f'(x)-2f'(0) \quad \cdots\cdots (7)$$

Putting $x=0$ in (7) and subtracting we get

$$f'(2\lambda+x)-f'(2\lambda)=f'(x)-f'(0)$$

or

$$f'(\mu+x)-f'(\mu)=f'(x)-f'(0) \quad \cdots\cdots (8)$$

where $\mu=2\lambda$. Replacing μ by $\mu+x$, $\mu+2x, \cdots, \mu+(n-1)x$ successively in (8) and adding we get

$$f'(\mu+nx)-f'(\mu)=n\{f'(x)-f'(0)\}$$

i. e. $$\operatorname*{Lim}_{n\to\infty} \frac{f'(nx)}{nx} = \frac{f'(x)-f'(0)}{x} \text{ (putting } \mu=0) \quad \cdots\cdots (9)$$

Now (9) shows that $\lim\limits_{n\to\infty} \dfrac{f'(nx)}{nx}$ exists for each $x \neq 0$. If x_1 and x_2 are two numbers such that $\dfrac{x_1}{x_2}$ is rational and positive, then we can find an infinite number of positive integers n and m so that $nx_1=mx_2$. Hence $\lim\limits_{n\to\infty} \dfrac{f'(nx)}{nx}$ is the same for all positive rational values of x. Hence there is a p such that

$$f'(x)=px+f'(0) \quad \cdots\cdots (10)$$

for positive rational values of x. Using this in (4) with $x=0$ we get

$$f(2\lambda)=f(0)+2\lambda\,[p\lambda+f'(0)]$$

or

$$f(\mu)=\frac{p}{2}\mu^2+\mu f'(0)+f(0) \quad \cdots\cdots (11)$$

for rational positive values of μ. Since $f(\mu)$ is continuous for all μ, (10) holds *for all positive* μ. This completes the proof in the case when $\mu>0$. Putting $\lambda=0$ in (4) we see that (10) is true when $\mu<0$. When $\mu=0$ (10) is obvious. Hence the theorem is proved.

Remark. The above proof looks a bit complicated and it is worthwhile finding a simple proof. If the second derivative exists for all x we get, by twice differentiating (4) and putting $x=0$, $f''(2\lambda)=f''(0)$, that is, $f''(x)$ is a constant. So, in this case, the proof is very simple. But in the above proof *no assumption* is made regarding the existence of the second derivative. It is also worthwhile finding out whether the theorem is true when $f(x)$ is defined and has a derivative in a given interval (a, b) and if θ in (1) is independent of x and h such that $x+h$ is in (a, b).

V. G. IYER.

REVIEWS

A History of the Conic sections and Quadric surfaces, By JULIAN LOWELL COOLIDGE, Ph.D.,D.Sc., LL D., Professor of Mathematics and Master of Lowell House (Emeritus) in Harvard University.— Oxford, Clarendon Press 1945. XI+214 pp. Price $ 6.

In general, mathematicians are not very history minded, otherwise the conics and quadrics would have found a comprehensive historical representation much earlier. In his history of elementary mathematics (famous for the reliability of historical statements), J. Tropfke has omitted the subject nearly completely, obviously because he did not consider it to be of a sufficiently elementary character. In the " Encyklopädie der mathematischen Wissenschaften", there are two valuable reports of Dingeldey (conics) and Staude (quadrics) which contain much valuable material and have also been quoted in the book under review. But those papers were written some 40 years back, more for reference than for continuous reading. A recent representation of the subject in the form of a monograph must therefore be welcomed. The subject of conics has been proved to be a perennial topic of Mathematics. In the second half of the 19th century, it looked as if after the work done by " the great projective school " and the progress of the theory of invariants, nothing important was left to be explored about conics. When in the seventies of that century, the Royal Academy in Brussels invited a competition for the best paper on the configuration of Pascal, the more progressive school of mathematicians may have considered this action to be out of date. As a matter of fact, the prize caused a flare-up of research on that particular subject, but it was rather an arrière-guarde action and not a great progress to new aims. The situation was completely changed, when Felix Klein discovered the astonishing fact, that algebraic investigations of Cayley admitted an interpretation in Non-Euclidean Geometry by which this branch of knowledge was completely reorganised. The magic wand by which the transformation was performed, is only a conic (or quadric). The same miracle succeeded for the second time, when Minkowski showed how easily the special theory of relativity can be interpreted as a Geometry in a 4-dimensional space where some (3-dimensional) quadric is distinguished. Afterwards the interest shifted to the infinitesimal quadrics, the quadric differential forms. Ricci calculus, tensor analysis etc. came out of the back waters of Mathematics into the lime light of general interest, but the genuine reason, why just the quadrics play such an important rôle, remained in the dark, till H. Weyl showed that the only Geometries where the affine connection is uniquely determined by the metric field, are those which can be defined by the help of quadratic forms. Whereas this trend of ideas was many-dimensional, investigations of another kind mainly developed in 2 dimensions at about the same time. In the affine differential Geometry and the theory of ovals (we may remember the names of Minkowski, Böhmer, Mukhopadyaya, Blaschke and his pupils), conics of various kinds play a leading part ; they bind together the Geometries " im kleinen " and " im grossen."

This is the stage of modern Mathematics, from which we look at the background of earlier historical development. Let us hope that when the romance of the conics and the quadratic forms in the last 50 years will be written, this thrilling story will find as able a historian, as the previous periods have got in Professor Coolidge. His book ranges from the earliest period of Greek Mathematics up-to the high days of the theory of invariants with a sparkling of later investigations. It makes an easy reading. The formal apparatus is restricted to the most necessary requirements and I suppose that college students will be able to follow. A great difficulty for historical works on Mathematics, especially when ancient times are concerned, is the choice of notations. When the author reproduces the investigations in their original notations, he will often bore the reader, whereas by the use of modern notations, the charm of the old works may get lost and what once upon a time was a great exploit, may look like a triviality. This dilemma is overcome in the present book in a masterly way. When describing the developement in the last few centuries, the author keeps an equal balance between the synthetic and the algebraic methods of treatment Little space is given to differential geometry ; topics which need higher analysis, are omitted. " When it comes to studying the lengths of arcs of conics, the story is very different. We run into elliptic integrals, a very large topic which we cannot enter into here". Systems of conics, conics in space and mechanical construction (but not the descriptive Geometry) of conics are also discussed. Especially these chapters contain much material which is not within general knowledge. The author never conceals his frank opinion and hereby a very refreshing and sympathetic note is added to the book. "There is a proof by infinitesimal geometry . which seems to me frankly, bad". With the same frankness, he also expresses his admiration : " the most stimulating book ever written on Geometry, to my way of thinking". The " Index of authors quoted " contains also the years of birth and death of those authors. Such biographical data are highly welcome in a historical book, which covers such a long range of time. A slight criticism may be permitted here. The readers might suppose that where the year of death is left blank, the person is still alive, but that is not so. Th. Reye and R. Sturm died in 1919, M. Cantor in 1920, M. Noether in 1921, Rosanes in 1922, W von Dyck in 1934 etc. The book contains fourteen chapters: I. The Greeks. II. Apollonius of Perga. III. Renewed interest in the conics. IV. The great projective school. V. The development of the algebraic treatment. VI. The introduction of new algebraic techniques. VII. Miscellaneous metrical theorems. VIII. Systems of conics. IX. Conics in space. X. Mechanical construction of conics. XI. Quadric surfaces, synthetic treatment. XII. Quadric surfaces, algebraic treatment. XIII. Quadric surfaces, higher algebraic treatment. XIV. Quadric surfaces, differential properties.—See also the review in " Mathematical Reviews ", Vol. 8, p. 1.

F. W. LEVI.

Ancient Indian Mathematics And Vedha, By L. V. GURJAR. Pp 202. Rs. 6/4 or 10 sh., 1947, (Ideal Book Service, 885 Sadasivpeth, Poona 2).

We welcome this nicely printed little brochure on the achievements of the ancient Hindus in Mathematics and Astronomy by an Indian Professor of Mathematics. The mathematics of the Sulva Sutras, Bhakshali Manuscript, Aryabhata, Brahmagupta, Mahavira and Bhaskara occupy two-thirds of the book and the methods of observation of heavenly bodies adopted by the early Hindus from the subject matter of the latter part. At the end of the book there is a good exposition of Jaisingh's beautiful method of stereographic projection which deserves to be taught our B SC, and B A., astronomy pupils. We strongly recommend the book for serious study by mathematics students in schools and colleges.

As the author draws heavily on translated texts, those who are conversant with the history of Hindu Mathematics are presented by European writers may find little that is new in this work except a few dubious interpretations of some Sanskrit verses. It is a pity that Prof. Gurjar should have completely ignored texts like Ganitakaumudi containing an exposition of magic squares, Yuktibhāsa with its approximations to π and the astronomical works Grahalaghava, Siddhanta Tatvaviveka and Siddhanta Sarvabhauma. Nor is there any excuse for failing to say anything about that Prince of Indian Astronomers, Varahamihira who gave us an encyclopaedia of all science, all art and all literature of his time (sixth century A. D.), in his Brihatsamhita. His Panchasiddhantika is of great historical and mathematical interest. There are important Indian contributions to trignometry, permutations and combinations, linear and quadratic interpolation, direct and inverse, literated methods of approximations which ought to find a place in any history of Indian mathematics.

Prof. Gurjar has unfortunately indulged too often in objectionable sentimental romanticism, while he has missed such important facts as that Bhaskara's chakravala method provides the shortest algorithm for the solution of the so called Pellian equation and that this method is not the same as Lagrange's solution but one which leads to a new type of canonical halfregular continued fraction discovered in the present century[1]. It is because of our ineffective expositions of Indian contributions that even very recently an American mathematician ventured to assert that Brahmagupta's and Bhaskara's works were empirical and tentative and their solution for the Pellian equation were merely lucky as they devoted much time to trival problems[2].

We have yet to evolve a first rate work on Indian mathematics and astronomy in which facts are stated and interpreted accurately and scientifically duly correcting all ignorant assertions made by some European historians. Meanwhile the book under review, the second of its kind, as a pioneer in

1 *Current Science*, VI, 12, June 1938, Pp. 602-604.
2 E. T. Bell. The Development of Mathematics. (1945) Pp. 98, 299.

the history of mathematics may have its own value as a land-mark in the evolution of our historico-mathematical development. We anxiously look forward to a more sober second edition with fuller content and a lower price so as to attract universal consumption in Independent India.

A. A. K.

Intermediate Geometry, BY M. V. JAMBUNATHAN, Seshadri & Co., Bangalore, 1946, Price Rs. 2/-, pp. 261.

College Geometry, BY KERKAR AND KANITKAR, Dastane Brothers Home Service, Poona, 1945, Price Rs. 1-8-0, pp. 144.

Plane Geometry for F. Y. Students, BY AGASHE AND CHANDRATREYA, Karnatak Publishing House, Bombay, 1945, Price Rs. 1-8-0, pp. 116.

Co-ordinate Geometry, BY H.D. BAGCHI AND B.N. MUKERJEE, U. N. Dhur & Sons, Ltd., Calcutta, Price Rs. 6/-, pp. 192

Analytical Solid Geometry, BY SANTI NARAYAN. Doaba House, Lahore, 1944, Price, Rs. 4/-, pp. 205.

The first three books in this list are text books in geometry the first intended for the Mysore Intermediate, and the second and third for the first year course in the Bombay University. They cover very nearly the same topics, the elements of solid geometry being an additional topic in the first book Prof. M. V. Jambunathan is an experienced teacher and author and knows where the young students wants help and guidance. Prof. Agashe and Chandratreya have been at special pains to explain fundamental notions and to lead up to the various circle constructions by a preliminary analysis. The printing and get up of both the books are good. The book by Kerkar and Kanitkar while covering the usual topics gives the reader the benefit of an index and an introduction to some additional topics like projection and complete quadrangle in the miscellaneous exercises.

Prof. Bagchi and B. N. Mukerjee's book is intended for the B. A. and B. Sc. pass classes of Indian Uuiversities. It gives a good introduction to the plane analytical geometry of the line, circle, and the conic sections in their standard forms. The book is well written and the printing and get up are exceedingly good.

Prof. Shanti Narayan's Analytical Solid Geometry is running through its third edition. The treatment extends up to the conicoids in their standard forms, (including parabaloids) and their plane and circular sections. There is a brief introduction to homageneous co-ordinates in 3-space, and to the imaginary circle at infinity but these ideas are not followed up in the later chapters. The book should be very useful for those to whom it is intended.

A. N. RAO.

ANNOUNCEMENTS AND NEWS

The Indian Mathematical Society has accepted the invitation of the Andhra University to held its fifteenth biennial conference at Waltair. The dates of the conference are the 22nd, 23rd and 24th of December 1947. Members wishing to attend the session are requested to write to Dr. T. Vijayaraghavan, Professor of the Mathematics, Andhra University, Waltair regarding their dates of arrival, and food requirments.

The following gentlemen have been admitted as members of the Society:—

1. T. H. Eidnani Esq., M.A., Lecturer in mathematics, c/o H. M. Eidnani, Britto Road, Karachi 5.
2. P. C. Joseph Esq., M.A., Assistant Professsr, St. Philomena's College, Mysore.
3. U. C. Sharma Esq. M.Sc., Lecturer, N. R. E. C. College, Bulandshaher, U. P.
4. N. A. Choudhary Esq., M.A., Lecturer, Muslim University, Aligarh.
5. B. K. Sen Cupta Esq, M.A., Headmaster, Srikhanda H. E. School, Srikhanda, Bengal.
6. R. P. Agnew Esq., Chairman, Department of Mathematics, Cornell University, Ithaca, N. Y. (U. S. A.)
7. H. B. Gupta Esq., M.A., Lecturer, Vaish College, Rohtak, Punjab.
8. G. C. Goela Esq., M A., Lecturer, Vaish College, Bhiwani District, Hissar, Punjab.

A. C. Choudhuri Esq., has been adimitted as a life member of the Society.

The Managing Committee of the Innian Mathematical Society has been reconstituted as follows:—

President	:	Prof. M. R. Siddiqi, Hyderabad.
Treasurer	:	Prof Ram Behari, Delhi.
Librarian	:	Prof. D. D. Kosambi, Bombay.
Secretaries	:	Prof. Vijayaraghavan, Waltair, and Prof. S. Mahadevan, Saidapet.
Other Members	:	Prof. N. R. Sen, Prof. R. Vaidyanathaswami, Prof. S. C. Dhar, Prof. A. C. Banerji, Prof. S. Chowla, Prof. F. W. Levi and Prof. C. N. Srinivasengar.

Dr. G. S. Mahajani, Principal, Fergusson College, Poona, has been appointed Vice Chancellor of the newly founded University of Rajaputana.

A new journal to be called *Simon Stevin* is to be published under the joint editorship of Dutch and Belgian mathematicians to replace *Christian Huygens*, *Mathematica B* and *Wis-en natuurkundig Tijdschrift*.

Prof. J. V. Uspensky of Stanford University died on January 27th 1947.

Prof. Edward B. Ross, who was for several years Professor of Mathematics at the Madras Christian College and a member of the Society, died in Edinburgh in January 1947. Sensitive and retiring by nature but warm and frank once the ice was broken, he was a keen mathematician and exerted a powerful influence on the mathematical education of those with whom he came into intimate contact.

Prof. A. C. L. Wilkinson, a former President of the Society (1917–1921) died in England in Feburary, 1947. Born in 1875, he was second Wrangler at Cambridge in 1898, took a first class in Tripos part II in 1899, and was also a Smiths Prizeman. He served in the Elphistone College and Deccan College, Bombay from which latter he retired as Principal in 1924. He was an active contributer to the newly started Journal of the Indian Mathematical Society and presided over the first three Conferences of the Society at Madras, Bombay and Lahore.

GLEANINGS

GEOMETRY

We agree not to define
Point, and plane, and single line.
Other terms and requisitions
We pin down with definitions.
We assume some postulates
As demand necessitates.
Given an hypothesis,
(Someone else can hand us this)
We proceed without confusion
To a logical conclusion,
Thus we prove a proposition
This the goal of our ambition.

K. O'BRIEN in *The Mathematics Teacher*, Jan. 1947.

Twinkle, twinkle, little star,
Now I know just what you are,
I can give you cast-iron facts
About your weight and parallax.
To me, your rate of radiation
Is simple as the earth's nutation.
Even if you're purest gas,
I can calculate your mass.
Think you're a mystery? What a hope!
Not with my new spectroscope!

The "*Sky and Telescope.*"

[Sent by K. S. B. SASTRI, Karachi.]